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FOUNDATIONS

Church, Alonzo, and Quine, W. V. Some theorems on definability and decidability. J. Symbolic Logic 17, 179–187 (1952).

$R(x, y)$ is a dyadic relation between pairs of natural numbers x and y. $H(x, y)$ means: (for suitable z and w)

$$y=x+1 \vee x=y+1 \vee y=x+2 \vee x=y+2 \vee x=6y \vee y=6x \vee [x=6z \& y=6w) \vee (x=6z+1 \& y=6w+1) \vee \{[(x=6z+1 \& y=6w+4) \vee (x=6w+4 \& y=6z+1)] \& R(z, w)]\}.$$

Then $x=y$ is equivalent to $(z)[H(x, z)=H(y, z)]$; further, formulae of pure quantification theory (q.t.) with H as their sole non-logical constant are obtained which are equivalent to $y=x+1$ and $R(x, y)$ respectively when the variables range over the natural numbers. The equivalences are established informally but hold in the usual systems of arithmetic, too, even when the quantifiers are given their finitist meaning. Since H is symmetric, the authors are able to show that there can be no (quasi-recursive) decision procedure for q.t. of symmetric relations: a current record. Q.t. of monadic predicates (properties) is decidable.

Elementary number theory is q.t. of the ternary relations $x=y+z$ and $x=y \cdot z$. Robinson [same J. 14, 98–114 (1949); these Rev. 11, 151] defined these relations by formulae of q.t. with the (dyadic) relations: divisibility $(x \mid y)$, succession $(y=x+1)$ and equality $(x=y)$. A single dyadic (symmetric) primitive for arithmetic is the H_0 obtained in the paper under review by taking $x \mid y$ for $R(x, y)$. Myhill [ibid. 15, 130 (1950); these Rev. 12, 233] has given an unsymmetric primitive. No attempt is made to give a simple system of axioms for the new primitive.

The result is the best possible of its kind since no set of monadic predicates can suffice for elementary number theory: proof from decidability of q.t. of properties, and undecidability of q.t. of relations. [In the reviewer's opinion the appeal to undecidability is artificial; for a stronger result holds: $a=b$ cannot be defined by properties, although q.t. of this relation is decidable. For, any formula $\phi(a, b)$ of q.t. with the non-logical constants $M_i(x)$ is equivalent in pure q.t. to a conjunction of formulae $A_j(a) \vee B_j(b) \vee L_j$, $1 \leqq j \leqq k$, where A_j, B_j are compounded of M_i and propositional connectives, and L_j contains neither a nor b. For some $a_1 < b_1 \leqq 2^k$, $A_j(a_1)=A_j(b_1)$ for all $j \leqq k$, since k properties cannot distinguish between 2^k+1 objects. Then $\phi(a_1, b_1)=\phi(b_1, b_1)$, but $b_1=b_1$ is true, $a_1=b_1$ is false. Actually neither $a=b+c$ nor $a=b \cdot c$ can be defined in terms of properties by formulae of q.t., even if identity is added.] *G. Kreisel* (Reading).

Craig, William, and Quine, W. V. On reduction to a symmetric relation. J. Symbolic Logic 17, 188 (1952).

$R_i(z_1 \cdots z_{d_i})$ are a sequence of relations or properties of natural numbers. p_i is the ith prime >1. $R(x, y)$ means $x \mid y$ or $x=0 \,\&\, y=p_0^{z_0} p_1^{z_1} \cdots p_{i+d_i}^{z_{d_i}} \,\&\, R_i(z_1 \cdots z_{d_i})$ holds. $H(x, y)$ is obtained from this $R(x, y)$ as described in the preceding review. Then each R_i can be defined with H as sole non-logical constant. (The R_i are not assumed to be definable in elementary number theory, so not necessarily definable in terms of H_0 of the preceding review.) *G. Kreisel.*

Rose, Alan. A formalisation of Post's m-valued propositional calculus. Math. Z. 56, 94–104 (1952).

Verf. gibt ein 10-gliedriges Axiomensystem für den m-wertigen Aussagenkalkül von Post [Amer. J. Math. 43, 163–185 (1921)] mit einem ausgezeichneten Wahrheitswert. Dann wird bewiesen, dass zu jeder Formel eine beweisbarkeitsgleiche konjunktive Normalform existiert. An die Stelle der Negation im 2-wertigen Kalkül treten hier iterierte Negationen und der T_m-Funktor von Rosser und Turquette [J. Symbolic Logic 10, 61–82 (1945); diese Rev. 7, 185]. *P. Lorenzen* (Bonn).

Markov, A. A. On unsolvable algorithmic problems. Mat. Sbornik N.S. 31(73), 34–42 (1952). (Russian)

The definition of "normal algorithms," and theorems about them, are repeated from the author's paper, Trudy Mat. Inst. Steklov. 38, 176–189 (1951) [these Rev. 13, 811]; and results on unsolvability, from the author's earlier papers [Doklady Akad. Nauk SSSR (N.S.) 77, 953–956; 78, 1089–1092 (1951); these Rev. 13, 4, 97], are restated in terms of "normal algorithms". *M. H. A. Newman.*

Vorob'ev, N. N. A constructive calculus of propositions with strong negation. Doklady Akad. Nauk SSSR (N.S.) 85, 465–468 (1952). (Russian)

Vorob'ev, N. N. The decision problem in the constructive calculus of propositions with strong negation. Doklady Akad. Nauk SSSR (N.S.) 85, 689–692 (1952). (Russian)

These two papers together contain the description, and an outline of the justification, of a decision algorithm for a certain calculus of propositions, Π^+, based on calculi of D. Nelson [J. Symbolic Logic 14, 16–26 (1949); these Rev. 10, 669] and A. A. Markov [Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 187–188 (1950)].

In the first paper Π^+ is specified. It has both "negation" ($\neg$) and "strong negation" ($\sim$) as primitives, as well as the usual $\vee$, &, $\supset$. There are 21 axioms, with "modus ponens" as the only rule of procedure. Two formulae, P and Q, are "equivalent" if $(P \supset Q)$ & $(Q \supset P)$ is provable in Π^+. A formula is "semi-reduced" if every occurrence of $\sim$ immediately precedes a variable; it is "reduced" if, further, $\neg$ does not occur; it is "fully reduced" if, further, it does not contain & nor any part $(P \vee Q) \supset R$. Theorem 5 states that, given any formula of Π^+, there can be found an equivalent formula which is a conjunction of fully reduced formulae.

Next a calculus is set up having a strong resemblance (noted by the author) to Gentzen's deduction-schemes [Math. Z. 39, 176–210 (1934)]. The objects in it are not logical formulae but "inferences", that is, words of the form $P_1, P_2, \cdots, P_k \vdash Q$, where the P_i are distinct semi-reduced

formulae (k can be 0), and Q is a semi-reduced formula or the empty word. An "initial inference" is one of the form $A \vdash A$ or $\sim A \vdash \sim A$, where A is a variable. An inference is "reduced" (or "fully reduced") if the terms P_i and Q are all reduced (or fully reduced) and Q is not empty. A "transformation scheme of inferences" (or, briefly, a scheme) is a figure

$$\frac{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_k}{\mathfrak{B}}$$

where $\mathfrak{A}_1, \cdots, \mathfrak{A}_k$, and $\mathfrak{B}$ are any inferences. A set of 13 "admissible" types of scheme is now listed, and a "proof by inference" is defined to be a finite sequence of inferences, $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_k$, each member, $\mathfrak{A}_i$, of which is either an initial inference, or such that

$$\frac{\mathfrak{A}_{i_1}\mathfrak{A}_{i_2}\cdots\mathfrak{A}_{i_k}}{\mathfrak{A}_i}$$

is an admissible scheme, with $i_1, i_2, \cdots, i_k < i$. It is announced (Theorem 7) that a formula P of the form $(P_1 \supset (P_2 \supset \cdots \supset (P_p \supset S))\cdots)$, where S is not of the form $X \supset Y$, is provable in Π^+ if and only if $P_1P_2^*\cdots P_p^* \vdash S$ is provable by inference (the stars indicating that duplicate P_i's are to be omitted).

This shews (since p can be 0) that a decision process for Π^+ can be derived from a decision process for provability by inference of fully reduced inferences, and the second paper is devoted to producing such a process. A number, $\chi(\mathfrak{A})$, is attached to each inference, and the algorithm takes the form of a method of replacing any fully reduced $\mathfrak{A}$ by another, of lower χ, provable by inference or not as $\mathfrak{A}$ is or is not, until an inference is reached for which provability can be decided by inspection. The argument depends on the precise form of the 13 "admissible schemes", and cannot be summarised. As an application of the method it is shewn that, if A is a variable, $\neg A \supset \sim A$ is not provable in Π^+.

Indications of proofs are sparse in the first paper, fuller in the second. *M. H. A. Newman* (Manchester).

***Novikov, P. S. On the uncontradictability of certain propositions of the descriptive theory of sets.** Trudy Mat. Inst. Steklov., v. 38, pp. 279–316. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

In 1940 Gödel published a proof that the axiom of choice and the generalized continuum hypothesis are consistent with the other axioms of set theory if these axioms themselves are consistent. [See his "The consistency of the continuum hypothesis" [Princeton, 1940; these Rev. 2, 66]. The result had been announced as early as 1938.] He did this by formulating these other axioms of set theory as a system Σ and then showing that Σ admits a model Δ for which the axioms in question are derivable in Σ. His result may be stated thus: The axioms in question are Σ-consistent. In the present paper the author similarly proves the Σ-consistency of certain propositions in the theory of analytic sets [see N. Lusin, Leçons sur les ensembles analytiques et leurs applications, Gauthier-Villars, Paris, 1930]. The theory of analytic sets can be incorporated in Σ simply by taking the fundamental Baire set I to be the class of all infinite sequences of natural numbers. By an ingenious and complicated correspondence between certain sets of Σ and certain elements of I the author is able to establish an isomorphism between a system represented in I and a portion of Gödel's model Δ. The propositions when Σ-consistency is established by this method include the following: the existence of a non-countable CA set not containing any perfect subset; the existence of a B_2 set not measurable in the sense of Lebesgue; and the existence of a function of class A_2 defined for all elements of I and discontinuous on every perfect set. (The author states that some of these propositions have been previously asserted without proof by Gödel, but he does not say where and the reviewer does not know.) A similar result concerning the separability of projective sets of higher classes is announced for later publication.

H. B. Curry (State College, Pa.).

Schröter, Karl. Deduktiv abgeschlossene Mengen ohne Basis. Math. Nachr. 7, 293–304 (1952).

Let X, Y be classes of statements of the predicate calculus. Y is a basis of X if (i) Y is independent, so that no statement $H \in Y$ is deducible from the class of statements $[Y - \{H\}]$, and (ii) every formula deducible from formulas in Y is also deducible from formulas in X. The class X is closed if every statement which follows from statements of X is itself in X. The following results are proved: (1) There is a class X such that X has a basis, but no class $Y \subset X$ is a basis of X; (2) there is a class X which has no basis; (3a) there are $2^{\aleph_0}$ classes X which are closed and have a basis; (3b) there are $2^{\aleph_0}$ classes X which are closed and have no basis. (3a) and (3b) form the main results of this paper. It is noted that (2) was announced without proof by Tarski [Monatsh. Math. Phys. 37, 361–404 (1930)]. *I. Novak Gál.*

Chinčin, A. Ja. The simplest linear continuum. Časopis Pěst. Mat. 76, 158–174 (1951). (Czech)

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 2(30), 180–197 (1949); these Rev. 11, 2.

Aleksandrov, A. D. Lenin's dialectics and mathematics. Časopis Pěst. Mat. 76, 237–250 (1951). (Czech)

Translated from Priroda 40, no. 1, 5–15 (1951).

Aleksandrov, A. D. On idealism in mathematics. Časopis Pěst. Mat. 76, 251–270 (1951). (Czech)

Translated from Priroda 40, no. 7, 3–11; no. 8, 3–9 (1951).

ALGEBRA

Rivier, W. Sur les jeux de combinaison et à propos d'un théorème d'Euwe. Rev. Gén. Sci. Pures Appl. 59, 197–210 (1952).

For its bearing on the rules of chess preventing indefinite play, a sequence is constructed, each term of which is one of m letters, and which is such that no three consecutive subsequences of n letters each are alike, thus generalizing a construction by the Dutch chess master Euwe.

J. Riordan (New York, N. Y.).

Thrall, R. M. A combinatorial problem. Michigan Math. J. 1, 81–88 (1952).

Soit m un entier positif et $(a) = (a_1, \cdots, a_k)$ une partition de $m = a_1 + \cdots + a_k$ en parties inégales $a_1 > \cdots > a_k > 0$. On fait correspondre à (a) un diagramme de a_1 lignes et de k colonnes, identique au graphe de Ferrers-Sylvester de (a), à ceci près qu'on a disposé les colonnes de façon que le sommet de la $j$ième colonne coïncide avec l'extrémité droite de la $j$ième ligne (les colonnes étant numérotées de gauche

à droite, les lignes de haut en bas). Si on remplace les m points du diagramme par les m nombres $1, \cdots, m$, on obtient un tableau. Parmi les $m!$ tableaux correspondant à un même diagramme (et donc à une même partition), ceux pour lesquels les nombres vont en croissant de gauche à droite dans chaque ligne, de haut en bas dans chaque colonne sont dits réguliers.

Dans ce travail, l'auteur établit que le nombre $g(a)$ de tableaux réguliers correspondant à la partition (a) a pour expression

$$g(a)=\frac{(a_1+\cdots+a_k)!\prod_{i<j}(a_i-a_j)}{a_1!\cdots a_k!\prod_{i<j}(a_i+a_j)}.$$

(L'évidente analogie avec l'expression $f(a)$ donnant le degré de la représentation irréductible du groupe symétrique $\mathfrak{S}_m$ correspondant à (a) a été utilsée par l'auteur pour la démonstration.) Le paragraphe final signale la liaison entre le cas particulier $(a)=(k, k-1, \cdots, 1)$ du problème précédent et un aspect de la théorie de la mesure en psychologie.

J. Riguet (Paris).

Shrikhande, S. S. On the dual of some balanced incomplete block designs. Biometrics **8**, 66–72 (1952).

The author proves that the dual design (i.e., the design obtained by interchanging the blocks and treatments) of a balanced incomplete block design [Yates, Ann. Eugenics **7**, 121–140 (1936)] with v^* treatments replicated r^* times in b^* blocks each containing k^* different treatments, so that every pair of treatments occurs in exactly λ^* blocks, is a partially balanced design defined by Bose and Nair [Sankhyā **4**, 337–372 (1939)] in the following two cases: (a) $\lambda^*=1$, (b) $r^*=k^*+2, \lambda^*=2$. Bose and Nair in the paper cited above [p. 371] note that in a number of actual examples the dual of a balanced incomplete block design happens to be partially balanced. Later Youden [Biometrics **7**, 124 (1951)] obtained the class of linked block designs by dualizing balanced incomplete block designs. The theorems proved in the paper thus give sufficient conditions for a linked block design to be partially balanced. *R. C. Bose.*

Norton, Donald A. Groups of orthogonal row-latin squares. Pacific J. Math. **2**, 335–341 (1952).

An n by n square of elements is a row (column) Latin square if each of its rows (columns) is a permutation of the same n distinct elements, say the integers $1, 2, \cdots, n$. If a row R_i consists of $a_1, \cdots, a_n$, we may associate it with the permutations $\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ a_1 & a_2 & \cdots & a_n\end{smallmatrix}\right)$ in the symmetric group S_n. Two row Latin squares A and B may be multiplied by multiplying the rows as elements of S_n. Here with $AB=C$ we have a row multiplication rule for row Latin squares. Row Latin squares are said to be orthogonal if, when superposed, all n^2 pairs (i, j) occur. It is shown that A and B are orthogonal row Latin squares if and only if there is a Latin square L with $AL=B$. From this, results on mutually orthogonal sets are found and in particular the result of H. B. Mann that if $A, A^2, \cdots, A^{n-1}$ are all Latin squares, then they are mutually orthogonal. *Marshall Hall* (Columbus, Ohio).

Janekoski, Viktor. Deux modes d'évaluation d'un déterminant de Gram. Bull. Soc. Math. Phys. Macédoine **2**, 131–137 (1951). (Macedonian. French summary)

Dans cette note on indique deux procédés vectoriels simples fournissant le développement du déterminant du troisième ordre du type de Gram. *Author's summary.*

Sagastume Berra, A. E. Two notes on the theory of determinants. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 199. Serie segunda, 19, Revista 4 (1951), 301–313 (1952). (Spanish. English summary)

The author considers an associative algebra A_n defined over a field K (with characteristic not 2) by the 2^n-1 units $e_1, \cdots, e_n, e_{12}, \cdots, e_{n-1,n}, e_{123}, \cdots, e_{n-2,n-1,n}, \cdots, e_{123\cdots n}$ (each e has r distinct indices $(r=1, 2, \cdots, n)$ arranged in natural order) with $e_{i_1i_2\cdots i_r}\cdot e_{j_1j_2\cdots j_s}=\operatorname{sgn}(i, j)e_{m_1m_2\cdots m_{r+s}}$, where $m_1, m_2, \cdots, m_{r+s}$ is the natural order arrangement of $i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s$ and

$$\operatorname{sgn}(i, j)=\operatorname{sgn}(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s)$$
$$=\operatorname{sgn}(p_1, p_2, \cdots, p_{r+s})=\operatorname{sgn}\prod_{\alpha>\beta}(p_\alpha-p_\beta),$$

$p_k=i_k$ $(k=1, 2, \cdots, r)$ $p_{r+t}=j_t$ $(t=1, 2, \cdots, s)$. If $\|\alpha_i^j\|$ is an nth order (square) matrix, $\alpha_i^j \in K$, then $a_i=\sum_{j=1}^n \alpha_i^j e_j$ $(i=1, 2, \cdots, n)$ are elements of A_n. Writing

$$a_{i_1}a_{i_2}\cdots a_{i_r}=\sum_{j_1<j_2<\cdots<j_r}\alpha_{i_1i_2\cdots i_r}^{j_1j_2\cdots j_r}e_{j_1j_2\cdots j_r},$$

the author defines the determinant formed by the $i_1, i_2, \cdots, i_r$ rows and $j_1, j_2, \cdots, j_r$ columns of $\|\alpha_i^j\|$ as the element $\alpha_{i_1i_2\cdots i_r}^{j_1j_2\cdots j_r}$ of K. All properties of determinants may be deduced from this definition. A determinant identity is derived and applied to reduce a quadratic form to a sum of squares. Two identities between Jacobians are established in the concluding part of the paper. *L. M. Blumenthal.*

Bucerius, H. Zur Theorie der linearen Gleichungen. Arch. Math. **3**, 103–107 (1952).

Familiar results on the solution of a system of linear equations are stated in a mixed tensor-vector notation as analogues of the integral representation of the solution of a self-adjoint boundary value problem for an inhomogeneous linear differential equation of the second order.

W. Givens (Knoxville, Tenn.).

***Denis-Papin, Maurice, et Kaufmann, A. Cours de calcul matriciel appliqué.** Editions Albin Michel, Paris, 1951. 304 pp. 1,600 francs.

This is an introduction to and popularization of matrix calculus, written for engineers. The elementary beginning, the helpful approach, the wealth of examples (algebraic, numerical, and engineering), the historical references, and the bibliography will undoubtedly appeal to engineers and numerically minded mathematics students, especially those teaching themselves.

Chapter 1 has definitions and basic theory of linear transformations, matrices, and quadratic forms. Chapter 2 deals with the diagonal form of matrices; non-linear divisors are not considered. Chapter 3 goes from the Cayley-Hamilton theorem to the Laplace transform of a matrix. In Chapter 4 matrices are applied to the study of general dynamical systems (conservative or not) with n degrees of freedom. Chapter 5 takes up matrices of four-terminal networks, and includes an 8-page table of networks. There are applications from the strength of materials in Chapter 6, while Chapter 7 describes some French punched-card and analogue machines. An appendix discusses determinants and zeros of polynomials, including the Routh and Hurwitz criteria for no zero to lie in the right half-plane. There is a 6-page bibliography.

The book is crowded with little misprints: e.g. (p. 299), "Routh E.-J. *Advanerd Rigid. Dynamies.*" Minor criticisms: p. 259: Cramer of Cramer's rule was Swiss, not

Swedish; p. 85: the identities are generally credited to Newton, not Bôcher. The reviewer (but possibly not the printer) was amused by the ideographic notation in which, for example, a square matrix a is written with a square around it: $\boxed{a}$. *G. E. Forsythe* (Los Angeles, Calif.).

***Parodi, Maurice. Sur quelques propriétés des valeurs caractéristiques des matrices carrées.** Mémor. Sci. Math., no. 118. Gauthier-Villars, Paris, 1952. 64 pp. 800 francs.

This is a leisurely résumé of selected recent researches on the localization of the characteristic values of a finite matrix in the complex plane. Let $A=(a_{ij})$ be of order n and have real or complex elements. Let $P_i=\sum_{j\neq i}|a_{ij}|$. The treatment starts from Hadamard's theorem that, if (*) $|a_{ii}|>P_i$ for all i, A is regular. In chapter I we read of generalizations by von Koch, Müller, Perron, Taussky, and the author. The principal business is in chapter II, which starts with Geršgorin's theorem (referred only to A. Brauer and Parodi) that the characteristic values of A lie in the union of the n disks $|a_{ii}-z|\leqq P_i$. Extensions by Brauer, Taussky, and the author are presented, together with applications to the stability of physical systems, the reducibility of polynomials, and the bounding of zeros of polynomials. In chapter III methods of Ostrowski are applied to such problems as learning how much the elements of a matrix A originally satisfying (*) can be varied, while preserving the regularity. Upper bounds are given for the real part of z, where z is any zero of the symmetric determinantal equation written $\|t_{ij}z^2+r_{ij}z+s_{ij}\|=0$. In chapter IV, following Wayland, similar determinantal equations are studied in which the polynomials are of order n. The bibliography of 33 titles, including 11 by the author, contains only memoirs whose results are actually used in this monograph. Even two references (pp. 25–26) to criteria of Routh, Hurwitz, and Franck are left unsupported by any bibliographical citation. One misses dozens of titles listed by Taussky [Bibliography on bounds for characteristic roots of finite matrices, Nat. Bur. Standards Rep. no. 1162 (1951)]. Several relevant papers by Ostrowski and others have appeared since the book was written.

Minor misprints noted: (1) in the second matrix on p. 8 the corner elements should be a_{n1}, a_{1n}, not b_{n1}, b_{1n}; (2) the summations in the second displayed formula on p. 22 should read like those in line 2 of p. 23. *G. E. Forsythe.*

Hestenes, M. R., and Karush, W. Solutions of $Ax=\lambda Bx$. J. Research Nat. Bur. Standards **47**, 471–478 (1951).

A, B are Hermitian matrices of order n with B positive definite. Let (x, y) denote the inner product of vectors x and y, and let $\mu(x)=(x, Ax)/(x, Bx)$. The stationary values of $\mu(x)$ are the characteristic roots λ_i of the determinantal equation (*) $|A-\lambda B|=0$. Gradient methods developed for $B=I$ by Kantorovič [Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948); these Rev. **10**, 380] and by the authors [J. Research Nat. Bur. Standards **47**, 45–61 (1951); these Rev. **13**, 283] are extended to find the λ_i. If G is a fixed positive definite matrix, define $\eta=\eta(x)=G^{-1}[Ax-\mu(x)Bx]$. Starting with x_0 arbitrary, a sequence $\{x_i\}$ is obtained by the recurrence $x_{i+1}=x_i-\alpha_i\eta_i$. Under mild restrictions on the real scalars $\{\alpha_i\}$ it is shown that $\mu(x_i)\downarrow\lambda$, some root of (*), and that every accumulation point of the x_i is a nonzero characteristic vector belonging to λ. If, moreover, λ is a simple root of (*), then x_i converges to a characteristic vector. Under the special hypothesis that $AG^{-1}B=BG^{-1}A$ it is proved that this λ is the least root of (*). There is a discussion of obtaining further characteristic roots and vectors, and of the possible equivalence of a general problem $|C-\lambda D|=0$ to some problem of type (*).

G. E. Forsythe (Los Angeles, Calif.).

de Rham, G. Sur un théorème de Stieltjes relatif à certaines matrices. Acad. Serbe Sci. Publ. Inst. Math. **4**, 133–134 (1952).

If the matrix G is real, has negative or zero nondiagonal elements and has no negative real characteristic value, then the elements of G^{-1} are all positive or zero. The proof is obtained by showing that for $0\leqq t\leqq 1$, $G_t=tG+(1-t)I$ transforms the boundary of the vector set $D=\{x; x_i>0\}$ into a set which cannot meet D. For $t=0$, $D\subset G_tD$ and hence by continuity $D\subset GD$ or $G^{-1}D\subset D$, which is equivalent to the desired conclusion. *W. Givens* (Knoxville, Tenn.).

Watanabe, Sigekatu, and Oyama, Koichi. On the minimal polynomial of a matrix. Rep. Univ. Electro-Commun. **1950**, no. 1, 51–70 (1950). (Japanese)

The set of all primitive idempotents of the algebra generated by a semi-simple complex matrix A over the complex field is called the spectral set of A. The authors generalize that notion of the spectral set to an arbitrary matrix and give a necessary and sufficient condition for a set of matrices to be the spectral set of some matrix. In the second part, the authors prove, by an elementary method, that every polynomial over a field k of characteristic 0 can be the minimal polynomial of some matrix with coefficients in k.

K. Iwasawa (Cambridge, Mass.).

Motzkin, T. S., and Taussky, Olga. Pairs of matrices with property L. Trans. Amer. Math. Soc. **73**, 108–114 (1952).

The following seven properties of pairs of square matrices of order n, say A and B with roots λ_i and μ_i, over an algebraically closed field of arbitrary characteristic are studied: 1) at least one root of $\alpha A+\beta B$ is of the form $\alpha\lambda_i+\beta\mu_i$ for fixed i and arbitrary α, β; 2) the roots of $\alpha A+\beta B$ are all given by $\alpha\lambda_i+\beta\mu_i$; 3) any function $f(A, B)$ has roots $f(\lambda_i, \mu_i)$, that is, A and B are simultaneously similar to triangular matrices; 4) A and B are k-commutative, that is, $B_k=0$ for some k, where $B_0=B$ and $B_{i+1}=AB_i-B_iA$; 5) $AB=BA$; 6) both A and B are functions of a matrix M; 7) A and B are co-canonical, that is, are simultaneously similar to Jordan normal forms with 1's at the same places above the main diagonal.

Known results are summarized and extended where necessary to establish the following relations between these properties: always a property implies its predecessor; for $n=2$, properties 1), 2), and 3) are equivalent and inequivalent to the equivalent 4), 5), 6), and 7); for $n\geqq 3$ all the properties are inequivalent; and for hermitian matrices (over the complex field) 2) through 7) are equivalent. On line 7 of page 112, the (1, 1) element of B should be -2 instead of $+2$. *W. Givens* (Knoxville, Tenn.).

Vivier, Marcel. Note sur les matrices extérieurement équivalentes. C. R. Acad. Sci. Paris **234**, 2327–2329 (1952).

Two p by q matrices θ and θ_1 are exterior equivalent if $(I_p\theta)=\zeta(I_p\theta_1)T$, where ζ is nonsingular and T is a permutation matrix. When T interchanges the k first columns of θ_1 with the corresponding columns of I_p, the factorization

$$\theta=\begin{pmatrix} I_k & 0 \\ -\xi & I_r \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} I_k & \phi \\ 0 & I_r \end{pmatrix} \quad \text{where} \quad \theta_1=\begin{pmatrix} x & \phi \\ \xi & y \end{pmatrix}$$

is obtained and some consequences noted. For a p by $p+q$ matrix $(M)=(\alpha\beta)$, the associated simple p-vector $[M]$ consisting of its minors of order p can be normalized by dividing each component by the determinant λ of the nonsingular p by p matrix α. Then $\lambda[I,\theta]=[M]$ and $\theta=\alpha^{-1}\beta$ is called a characteristic matrix of $[M]$ associated with λ. Two characteristic matrices are exterior equivalent.

W. Givens (Knoxville, Tenn.).

Fréchet, Maurice. Les solutions non commutables de l'équation matricielle $e^X\cdot e^Y=e^{X+Y}$. Rend. Circ. Mat. Palermo (2) **1**, 11–27 (1952).

Matrices of order two such that

(1) $$XY\neq YX \quad \text{but} \quad e^Xe^Y=e^{X+Y}=e^Ye^X$$

are studied. [It is erroneously stated ". . . en tout cas que si $e^Xe^Y=e^{X+Y}$ on a $e^Xe^Y=e^Ye^X$." A counter example is furnished by

$$X=\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix},\quad Y=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

if α is any non-zero root (an infinite number of which do exist) of $(1-s)e^s=1$, for then $e^X\cdot e^Y=e^{X+Y}\neq e^Y\cdot e^X$.] It is proved that the only solutions $X=A$, $Y=B$ of order two of (1) satisfy: $e^A=\nu I$, $e^B=\nu' I$ and A, B, and I are linearly independent. A complete (parametric) solution is given and the real and complex cases are distinguished.

W. Givens (Knoxville, Tenn.).

Toscano, Letterio. Calcolo degli elementi della potenza di una matrice del terzo ordine. Anais Fac. Ci. Porto **35**, 14–27 (1951).

Using the fact that a matrix of order three satisfies an equation $x^3-px^2+qx-r=0$, explicit expressions for A^n as a linear combination of I, A, and A^2 are found in terms of the invariants and also in terms of the roots. The recursion formulas $x^{(i+1)}=Ax^{(i)}$ are solved to express the components of $x^{(n)}$ in terms of those of $x^{(0)}$ to obtain again A^n.

W. Givens (Knoxville, Tenn.).

Kurepa, Djuro. Remarque sur le produit de deux nombres complexes du même signe. Acad. Serbe Sci. Publ. Inst. Math. **4**, 176–177 (1952).

Correction of an error in a previous paper [same Publ. **3**, 89–99 (1950); these Rev. **12**, 581]; and description of two different orderings in the complex number field.

E. Hewitt (Seattle, Wash.).

Urazbaev, B. M. On p-adic expansions of the roots of two-term equations in the case when the degree is divisible by p. Izvestiya Akad. Nauk Kazah. SSR. **60**, Ser. Mat. Meh. **3**, 106–132 (1949). (Russian. Kazak summary)

A careful study is made of the p-adic expansions of the roots of the equation $x^n-a=0$ for the case n divisible by p. The expansions obtained are of the form $x=\sum\lambda_i p^{\rho_i}$, where the λ_i are prime to p and the ρ_i are increasing rational numbers. In the case under discussion the ρ_i are actually bounded and arbitrary high powers of p occur in their denominators. The discussion is broken up into several cases and a number of numerical examples are given.

W. H. Mills.

Abstract Algebra

Bourbaki, N. Éléments de mathématique. XIV. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre VI: Groupes et corps ordonnés. Chapitre VII: Modules sur les anneaux principaux. Actualités Sci. Ind., no. 1179. Hermann et Cie, Paris, 1952. ii+159+iii pp. 1800 francs.

Chapter 6 begins with generalities on partially ordered semigroups and groups, which are rapidly specialized to the lattice-ordered (réticulé) case. Application is made to divisibility in integral domains. Uniqueness of factorization is proved for lattice-ordered groups with descending chain condition. Then ordered fields are taken up. The theorem of Artin-Schreier (a field can be ordered if -1 is not a sum of squares) is proved after the manner of Serre [C. R. Acad. Sci. Paris **229**, 576–577 (1949); these Rev. **11**, 76]. In the text everything is commutative, but the exercises occasionally dip into the non-commutative and even non-associative. Other highlights of the exercises are a close scrutiny of the Chinese remainder theorem, and the embedding of a lattice-ordered group into a direct product of simply ordered groups.

Chapter 7 is devoted to modules over a principal ideal ring R (which in the text means commutativity and no divisors of 0). After the appropriate considerations on factorization in R, the decomposition of a torsion R-module into primary modules is established. As an application, the partial fraction decomposition in the quotient field of R is derived. Free R-modules are briefly considered; one notes the use of well-ordering in proving that a submodule of a free module is free. Next comes the finitely generated case. Uniqueness of a decomposition into cyclic modules is proved over any commutative ring, by the technique of the exterior product of modules. The decomposition itself is proved by studying the pair consisting of a finitely generated free module and a submodule. The application to matrices is given in considerable detail, and there is an appendix establishing the connection between modules and matrices by the use of Kronecker products. The volume concludes with an historical note on the two chapters. The exercises range over many topics, such as: principal ideal rings with divisors of 0, Euclidean algorithm and units in quadratic fields, perfect numbers, the weak form of the prime number theorem, duality of modules, and a long series of exercises leads all the way to Ulm's theorem. In a later edition, M. Bourbaki may wish to change the hint for ex. 10 on p. 84; the proof by Baer [Duke Math. J. **3**, 68–122 (1937), p. 121] is neater and does not need more than one prime.

I. Kaplansky (Chicago, Ill.).

Katětov, M. Remarks on Boolean algebras. Colloquium Math. **2** (1951), 229–235 (1952).

The author constructs a denumerable normal space P in which every point is the limit of some non-constant sequence, and such that P allows only one homeomorphism, the identity. He observes that its Stone-Čech compactification B is 0-dimensional and likewise has only one homeomorphism. The Boolean lattice of open closed sets in B thus provides an answer to Problem 74 [G. Birkhoff, Lattice theory, rev. ed., Amer. Math. Soc. 1948, p. 162; these Rev. **10**, 673]. Turning to Problem 62 [op. cit., p. 62], he shows that a Boolean lattice is a Hausdorff space in its interval topology precisely when it is atomic.

R. Arens.

Rieger, Ladislav. Some remarks on automorphisms of Boolean algebras. Fund. Math. **38**, 209–216 (1951).

The author constructs a Boolean algebra that has no proper homomorphism onto itself, thereby solving Problem 74 in G. Birkhoff's Lattice theory [Amer. Math. Soc. Colloq. Publ., v. 25, 2d ed., New York, 1948; these Rev. **10**, 673]. He grants priority to Katětov, referring to the paper reviewed above. The problem was also solved recently by B. Jónsson [Proc. Amer. Math. Soc. **2**, 766–770 (1951); these Rev. **13**, 201]. *P. R. Halmos* (Chicago, Ill.).

Löwig, Henry. On transitive Boolean relations. Czechoslovak Math. J. **1(76)** (1951), 199–201 (1952) = Čehoslovack. Mat. Ž. **1(76)** (1951), 225–228 (1952).

Given elements a, b, c, d of a Boolean algebra, set xRy if and only if $axy+bxy'+cx'y+dx'y'=0$. Then R is transitive if and only if $a(b+c)d\neq 0$ or $a+d\leqq b+c$. In addition, R is a quasi-ordering if and only if $a=d=0$, an equivalence relation if and only if $a=d=0$ and $b=c$, a partial ordering if and only if $a=d=0$ and $b+c=1$, a lattice ordering (equivalently, a partial ordering determining a Boolean algebra) if and only if $a=d=0$ and $b=c'$. *P. M. Whitman.*

Četković, Simon. Résolution d'un système infini d'équations d'ensembles. Bull. Soc. Math. Phys. Serbie **4**, nos. 1–2, 51–59 (1952). (Serbo-Croatian. French summary)

Soit $a_1, a_2, \cdots$ une suite d'éléments d'une algèbre de Boole. On notera x' le complémentaire de x. Pour que la suite $x_1, x_2, \cdots$ soit solution du système infini d'équations $(x_i \cap a'_{i+1}) \cup (x'_{i+1} \cap a_i) = a_i$ $(i=1, 2, \cdots)$ il faut et il suffit que x_i soit de la forme $x_i = m_i \cup n_i \cup n_{i+1}$ où m_i et n_i sont deux suites satisfaisant à

$$m_i \leqq (a_i \cup a_{i+1}) \cap a'_{i+1}, \quad n_i \leqq a_{i+1} \cap a_{i-1} \cap a_i' \cap a'_{i-2}.$$

(L'auteur suppose que l'algèbre de Boole est constituée par l'ensemble des parties d'un ensemble mais une modification évidente de la démonstration montre que l'énoncé est valable pour une algèbre de Boole quelconque.) Parmi les erreurs typographiques, signalons que dans la formule (3.9) de la p. 57 le signe $\subseteq$ soit être renversé. *J. Riguet.*

Frink, Orrin. A proof of the maximal chain theorem. Amer. J. Math. **74**, 676–678 (1952).

A simply ordered subset of a partially ordered set is called a chain. The maximal chain theorem due to Hausdorff states that every chain in a partially ordered set is contained in a maximal chain. This theorem is well-known to be equivalent to Zorn's lemma, and, like the latter, it owes its great usefulness to the fact that it allows one to avoid the theory of ordinal numbers and well-ordered sets in a great variety of mathematical proofs. For this reason it is desirable and useful to deduce this theorem in a simple way directly from the axiom of choice, without making use of ordinal numbers or of Zermelo's theorem on well-ordering. In fact, several proofs were published, especially in recent years, which deduce the maximal chain theorem in such a manner. However, all proofs in the literature, either of this theorem or of Zorn's lemma, involve the notion of well-ordering. The present proof of the author is the first one which does not involve the concept of a well-ordered set.

The author's very simple and elegant proof for the maximal chain theorem was suggested by Zermelo's second proof of the well-ordering theorem, and its fundamental idea is the following. If the theorem is false, then in some partially ordered set P there is a chain A not contained in any maximal chain. Now, corresponding to each chain C which includes A as a subset we select, by means of the axiom of choice, a larger chain C', called the successor of C, containing only a single element of P not in C. A collection K of chains of P all of which include A is said to be complete if A is in K, the successor of each member of K is in K, and K contains the union of each chain of its chains. Then the intersection of all complete collections of chains in P can be proved to be a chain which implies an obvious contradiction. *T. Szele* (Debrecen).

Higman, Graham. Ordering by divisibility in abstract algebras. Proc. London Math. Soc. (3) **2**, 326–336 (1952).

A set with a reflexive, transitive order relation has the finite basis property (f. b. p.) in case every dual ideal (= subset S such that x in S and $y \geqq x$ imply y in S) is finitely generated (consists of $x_1, \cdots, x_n$ and all $y \geqq x_i$ for some i). The author considers abstract algebras with finitely many finitary operations, having such an order relation compatible with the algebra operations and satisfying $\mu(x_1, \cdots, x_n, a, y_1, \cdots, y_m) \geqq a$ for all operations μ and all a, x_i, y_j in the algebra. He proves such an algebra has the f. b. p. if any set of (algebra-) generators has. This theorem solves a problem on multiplicative semigroups of positive integers proposed by Erdös [Amer. Math. Monthly **56**, 480 (1949)]. Besides making other combinatorial applications, the author uses this theorem to give another proof that the power series with exponents in a linearly ordered quasigroup form a (not necessarily associative) ring provided the coefficients are from a (not necessarily associative) ring. This last is proved as a culmination of a more general discussion of the analog of a power series ring, with exponents in any ordered groupoid (= partially ordered algebra with one binary operation) with cancellation.

The paper also contains a more general theorem than that above, which proves the f. b. p. for certain algebras with infinitely many operations. As an example of many possible applications, the author proves this theorem about arbitrary groups G: Consider the algebra of subgroups of G with operations $H, K \to [H, K]$ (commutator) and $H \to nH$ (subgroup generated by the nth powers of elements of H) for n in a fixed set of positive integers, I. Let U_0 be the subalgebra generated by G. If I has no infinite subset of mutually indivisible integers, then the ascending chain condition holds for subgroups of G which are unions of elements of U_0. *D. Zelinsky* (Evanston, Ill.).

Thurston, H. A. A note on continued products. J. London Math. Soc. **27**, 239–241 (1952).

D'après la terminologie introduite dans un précédent travail [même J. **24**, 260–271 (1949); ces Rev. **11**, 316] une opération $(\nu+1)$-aire, notée par une parenthèse, est dite '$p=p+q$' partiellement associative lorsque l'on a identiquement

$$(x_1 \cdots x_{p-1}(x_p \cdots x_{p+\nu})x_{p+\nu+1} \cdots x_{2\nu+1}) = (x_1 \cdots x_{p+q-1}(x_{p+q} \cdots x_{p+q+\nu}) \cdots x_{2\nu+1}).$$

Elle est dite (j, k)-associative si elle est '$p=p+q$' partiellement associative lorsque $q=0$ ou lorsque p est un multiple de j et q un multiple de k (k étant un multiple de j et ν un multiple de k) et seulement dans ce cas.

Si $a_1, \cdots, a_r$ sont des nombres entiers satisfaisant à (C) $a_1 \leqq \nu$, $a_2 \leqq 2\nu, \cdots, a_r \leqq r\nu$, $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_r$, l'auteur désigne par $(a_1, \cdots, a_r)$ l'expression suivante (qu'il appelle produit continu)

$$(x_0 \cdots x_{a_1-1}(x_{a_1} \cdots x_{a_2-1}(x_{a_2} \cdots (x_{a_r} \cdots x_{a_r+\nu}) \cdots x_{(r+1)\nu})$$

les $r-1$ demi parenthèses ")" absentes devant être disposées de façon convenable (unique) pour donner un sens. Il montre ici qu'étant donné une opération (j, k)-associative, si $a_1, \cdots, a_r$ sont des multiples de j satisfaisant à la condition (C), s'il en est de même pour $b_1, \cdots, b_r$, et s'il existe une permutation σ des r premiers entiers telle que, quel que soit $i \leqq r$, $a_i \equiv b_{\sigma(i)} \pmod{k}$, alors $(a_1, \cdots, a_r) = (b_1, \cdots, b_r)$.

J. Riguet (Paris).

Thurston, H. A. The structure of an operation. J. London Math. Soc. **27**, 271–279 (1952).

Etant donné k entiers positifs $n_1, \cdots, n_k$, une opération $(n_1+\cdots+n_k)$-aire, notée /, sera dite suivant la terminologie de l'auteur produit en parallèle des k opérations n_i-aires $*_i$ $(i=1, \cdots, k)$ par l'opération k-aire $*$ lorsque

$$/x_1 \cdots x_{n_1} \cdots x_{n_1+n_2} \cdots x_{n_1+\cdots+n_k} = *(*_1 x_1 \cdots x_{n_1})(*_2 x_{n_1+1} \cdots x_{n_1+n_2}) \cdots \times (*_k x_{n_1+\cdots+n_{k-1}+1} \cdots x_{n_1+\cdots+n_k}).$$

Alors si les opérations /, $*$, $*_1$, $\cdots$, $*_k$ sont définies sur l'ensemble S, pour que / soit le produit en parallèle des $*_i$ par l'opération $*$ il faut et il suffit que pour tout $i \leqq k$, la puissance de $S^{n_i}/\mathfrak{Q}_i$ soit inférieure ou égale à celle de S, $\mathfrak{Q}_i$ étant la relation d'équivalence sur S^{n_i} définie par: "$a \equiv b \bmod \mathfrak{Q}_i$ ⇄ quels que soient $x \in S^{n_1+\cdots+n_{i-1}}$, $y \in S^{n_{i+1}+\cdots+n_k}$, $/xay = /xby$". On a une définition et un énoncé analogue pour les produits en série. Le reste du travail est consacré à l'examen de cas particuliers (notamment celui pour lequel $*_1 = \cdots = *_k$).

J. Riguet (Paris).

Baer, Reinhold. Kriterien für die Existenz eines Einselements in Ringen. Math. Z. **56**, 1–17 (1952).

Let R be a ring with Perlis-Jacobson radical N. If R/N has an identity and in R there exists an idempotent element which corresponds to the identity of R/N, then the following are equivalent. (1) Each such idempotent element is a left identity of R. (2) There exists in R a left identity element. (3) For each element x in R, $x \in Rx$. (4) If $Rx = Nx$, then $x = 0$. (5) $Z = RZ$ for each two-sided ideal Z in R; and if $x \in N \times N$, then $x \in Rx$ or $x \in xR$. (6) Each right ideal of R (other than R) is contained in a maximal right ideal and N is the intersection of all maximal right ideals. (7) If J is a right ideal such that $N+J=R$, then $J=R$. If, now, we assume only that R/N has an identity, the following are equivalent. (1) R has a left identity. (2) For each $x \in R$, $x \in Rx$; and $R = N+J$ for a right ideal J implies that $J=R$. (3) For each $x \in R$, $x \in Rx$; and if $R = N+yR$, then $y \in yR$. (4) $Rx = Nx$ implies that $x=0$; and if $R = N+yR$, then $y \in yR$. (5) If $Rx = Nx$, then x is nilpotent; and $R = N+J$ for a right ideal J implies that $J=R$. A number of other related results are obtained. If A is a subset of a ring R, let $A_R = \{x \in R; Ax = 0\}$, and $A_L = \{x \in R; xA = 0\}$. The ring R is autodual if for each right ideal J and each left ideal Q, $J = J_{LR}$ and $Q = Q_{RL}$. The author obtains a number of conditions, each of which is equivalent to the existence of an identity in an autodual ring.

N. H. McCoy.

Suškevič, A. K. On a linear algebra of infinite order. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 119–126 (1950). (Russian)

The object under study (generally called a ring rather than an algebra) is the ring of all formal power series with integral coefficients. Sundry topics concerning factorization and ideal theory in this ring are examined, but the author does not discuss uniqueness of factorization. There is no reference to the abundant literature on the subject, which begins with the paper of Cahen [C. R. Acad. Sci. Paris **152**, 124–127 (1911)].

I. Kaplansky (Chicago, Ill.).

Pickert, Günter. Zwischenkörperverbände endlicher inseparabler Erweiterungen. Math. Z. **55**, 355–363 (1952).

Problem: Given a finite algebraic extension L of an arbitrary field K. When does the set of intermediate fields form a modular lattice under intersection and composition? Author's answer: When L/K is purely separable, this is equivalent to a similar problem in group theory. When L/K is pure inseparable, a lattice is modular if and only if the multiplicity of L/K is less than or equal to 2. An example due to S. MacLane [Duke Math. J. **4**, 455–468 (1938)] is an essential part of this proof. The last part of the proof (p. 358) requires some corrections which the reader can easily supply. If L/K is arbitrary, then a lattice is modular if and only if (a) this is true for S/K where S is maximal separable (over K) in L; (b) multiplicity $\leqq 2$; and (c) every intermediate field of multiplicity >1 contains S. An example shows these conditions can be satisfied with multiplicity 2 and $S \neq K$. The author uses earlier work of his on inseparability [Math. Z. **52**, 81–136 (1949); **53**, 133–135 (1950); these Rev. **11**, 313; **12**, 316]. ~~Some of the proofs are overcomplicated.~~

G. Whaples (Bloomington, Ind.).

Dieudonné, Jean. Les extensions quadratiques des corps non commutatifs et leurs applications. Acta Math. **87**, 175–242 (1952).

Chapter I. A complete determination of the galoisian extensions K of a (not necessarily commutative) field L such that K is of rank (either left or right, and consequently both) 2 over L. Chapter II. Let L be a field, $\xi \to \bar{\xi}$ an involutive antiautomorphism of L, σ_i an automorphism of L such that σ_i^2 is inner $(0 \leqq i \leqq m-1)$ and $\xi \to \bar{\xi}$, $\sigma_0, \cdots, \sigma_{m-1}$ commute in pairs up to inner automorphisms, ψ_0 an involutive correlation relative to $\xi \to \bar{\xi}$ of a finite-dimensional right vector space E_0 (i.e., a mapping of E_0 onto its dual E_0^*, which is semilinear relative to $\xi \to \bar{\xi}$ and equals its own transpose), u_i an involutive collineation of E_0 relative to σ_i (i.e., a mapping of E_0 onto E_0 which is semilinear relative to σ_i with square a scalar multiplication) $(0 \leqq i \leqq m-1)$, and suppose that $\psi_0, u_0, \cdots, u_{m-1}$ are projectively permutable (permutable up to a scalar factor) in pairs. It is shown, using chapter I, how the study of the group of all collineations of E_0 which are projectively permutable with $\psi_0, u_0, \cdots, u_{m-1}$ can be reduced to the study of the group of all collineations which are projectively permutable with a single involutive correlation or collineation, the field L, however, being replaced by another field obtainable from L either by a succession of quadratic extensions or by a succession of quadratic restrictions. The preceding results are used in chapter III to complete results of van der Waerden [Gruppen von linearen Transformationen, Springer, Berlin, 1935] on the orthogonal groups, and in chapter IV and an appendix to certain questions concerning the classical groups which were unanswered or inadequately answered in previous publications by the author [Sur les groupes classiques, Hermann, Paris, 1948; Summa Brasil. Math. **2**, no. 6, 59–94 (1950); Mem. Amer. Math. Soc., no. **2** (1951); these Rev. **9**, 494; **13**, 531]. The details are too gruesome to list here.

E. Kolchin.

Kasch, Friedrich. Über den Satz vom primitiven Element bei Schiefkörpern. J. Reine Angew. Math. **189**, 150–159 (1951).

Let L be a finite Galois extension of a division ring H, and let T be the centralizer of H in L (so that T is the divi-

sion algebra over the center Z of L which is associated with the (closed) Galois group of L over H). Theorem: If the center of T is separable over Z, then $L=H(a, a')$, where a and a' are conjugate under an inner automorphism of L. The well-known theorem of the primitive element for finite separable (commutative) extension fields is an immediate corollary. The proof of the theorem relies on consideration of a maximal division subring $M=H(a)$ without inner automorphisms over H. It is remarked in a footnote that, in Albert's theorem on two-element generation of a separable algebra [Bull. Amer. Math. Soc. 50, 786–788 (1944); these Rev. 6, 115], the two generators may be taken to be conjugate under an inner automorphism. *R. D. Schafer.*

Nakayama, Tadasi. Galois theory of simple rings. Trans. Amer. Math. Soc. 73, 276–292 (1952).

L'auteur étend aux anneaux simples satisfaisant à la condition minimale la théorie de Galois des algèbres simples développée par Hochschild [mêmes Trans. 69, 292–301 (1950); ces Rev. 12, 315]. Un groupe d'automorphismes Φ d'un tel anneau A est dit régulier si: 1) l'anneau T_Φ engendré par les éléments de A qui définissent les automorphismes intérieurs contenus dans Φ est simple et de rang fini sur le centre Z de A; 2) Φ contient le groupe Φ_0 des automorphismes intérieurs définis par les éléments de T_Φ, et $(\Phi:\Phi_0)$ est fini; $(\Phi:\Phi_0)[T_\Phi:Z]$ est alors appelé l'ordre réduit de ϕ. Un sous-anneau simple C de A est dit faiblement (resp. fortement) normal si son commutant dans l'anneau $\mathfrak{A}$ des endomorphismes du groupe additif A est engendré (par rapport à l'anneau A_l des multiplications à gauche dans A) par des applications semi-linéaires (resp. des diautomorphismes) du A_l-module A. Cela étant, l'auteur démontre les théorèmes suivants: I) Si Φ est régulier, l'anneau $I(\Phi)$ des éléments de A invariants par Φ est un anneau simple fortement normal dans A, et le degré $[A:B]$ (à gauche ou à droite est fini et égal à l'ordre réduit de Φ; en outre le commutant $V_A(I(\phi))$ de $I(\Phi)$ dans A est l'anneau simple T_Φ, et Φ est le groupe d'automorphismes de A laissant invariants les éléments de $I(\Phi)$. II) Si B est un sous-anneau simple de A, faiblement normal, tel que $[A:B]$ soit fini et que le commutant $V_A(B)$ de B dans A soit simple, alors le groupe Φ des automorphismes de A laissant invariants les éléments de B est régulier, et B est identique à l'anneau des éléments de A invariants par Φ. Il résulte de I et II qu'il y a correspondance biunivoque entre les sous-groupes réguliers d'un groupe régulier Φ, et les sous-anneaux simples de A contenant $I(\Phi)$ et ayant des commutants simples dans A. III) Si B et B' sont deux tels sous-anneaux, et s'il y a un isomorphisme de B sur B' laissant fixes les éléments de $I(\Phi)$, cet isomorphisme peut s'étendre à un automorphisme de A appartenant à Φ. Les méthodes de l'auteur reposent essentiellement sur le lemme suivant: si C est un sous-anneau faiblement normale de A, A est complètement réductible en tant que bimodule sur A (à gauche) et C (à droite).

J. Dieudonné (Ann Arbor, Mich.).

Nakayama, Tadasi. Orthogonality relation for Frobenius- and quasi-Frobenius-algebras. Proc. Amer. Math. Soc. 3, 183–195 (1952).

The paper contains a new proof for the orthogonality relations for the coefficients of the representations of Frobenius algebras. These relations, which generalize Schur's relations for the representations of groups, have been discovered by the author [see Nesbitt and Thrall, Ann. of Math. (2) 47, 551–567 (1946); these Rev. 8, 64; and Brauer, Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, pp. 233–245, Füssli, Zürich, 1945; these Rev. 7, 238]. At the same time, the author gives an extension of these relations to the case of quasi-Frobenius algebras.

R. Brauer (Cambridge, Mass.).

Osima, Masaru. A note on symmetric algebras. Proc. Japan Acad. 28, 1–4 (1952).

C. Nesbitt and W. M. Scott have shown that an algebra over an algebraically closed field is symmetric if and only if its basic algebra is symmetric [Ann. of Math. (2) 44, 534–553 (1943); these Rev. 5, 89]. The author gives a new proof. He shows further that a uniserial algebra over an algebraically closed field is symmetric. *R. Brauer.*

Satake, Ichirô. On the structure of Brauer group of a discretely-valued complete field. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 1, 1–10 (1951).

T. Nakayama [J. Reine Angew. Math. 178, 11–13 (1937)] proved that if k is complete under a discrete valuation, then the group of those division algebras with center k whose residue class algebras are separable over the residue class field $\mathfrak{k}$ of k is the direct product of two subgroups; the first consisting of unramified algebras and isomorphic to the group of all division algebras with center $\mathfrak{k}$, and the second consisting of ramified algebras and dual to the Galois group of the maximal abelian extension of k. This generalized the proof already given by Witt [ibid. 176, 153–156 (1936)] for the case when $\mathfrak{k}$ has no inseparable extensions. This paper gives a new proof of these results. A principal feature of it is a formalism for expressing cyclic algebras in terms of characters of the Galois group which was independently discovered by the reviewer [Ann. of Math. (2) 55, 367–372 (1952); these Rev. 13, 720]. *G. Whaples.*

Kametani, Shunzi. An elementary proof of the fundamental theorem of normed fields. J. Math. Soc. Japan 4, 96–99 (1952).

The theorem is: A normed field over the complex numbers is isomorphic to the field of complex numbers.

M. Jerison (Lafayette, Ind.).

Skornyakov, L. A. Alternative fields of characteristic 2 and 3. Ukrain. Mat. Žurnal 2, no. 3, 94–99 (1950). (Russian)

Supplementing an earlier paper [same Žurnal 2, 70–85 (1950); these Rev. 12, 668] the author shows that an alternative (but not associative) division ring of characteristic 3 is a Cayley-Dickson algebra over its center. This is in agreement with the Bruck-Kleinfeld result [Proc. Nat. Acad. Sci. U. S. A. 37, 88–90 (1951); these Rev. 13, 8] which covers all characteristics $\neq 2$.

More recently Kleinfeld has shown [ibid. 37, 818–820 (1951); these Rev. 13, 527] that even for characteristic 2 an alternative division ring is associative or is a Cayley-Dickson algebra over its center. But in this paper Skornyakov claims to show that every alternative division ring of characteristic 2 is associative. This result appears to be false. Consider the field $F=F(\alpha, \beta, \gamma)$ obtained by adjoining three indeterminates α, β, γ to the field with two elements. Then it can be shown that in F there do not exist elements $\lambda, \mu, \rho, \sigma$ such that

$$\gamma=\lambda^2+\lambda\mu-\alpha\mu^2-\beta\rho^2-\beta\rho\sigma+\alpha\beta\sigma^2.$$

This may be established by a study of the degrees of $\lambda, \mu, \rho, \sigma$ as rational functions in first γ, then β, and then α. But this condition is that given by Schafer [Bull. Amer. Math.

Soc. 49, 549–555 (1943); these Rev. 5, 33] for there to exist a Cayley-Dickson division algebra over F, which is of characteristic 2. *Marshall Hall* (Columbus, Ohio).

Skornyakov, L. A. On the theory of alternative fields. Uspehi Matem. Nauk (N.S.) 5, no. 5(39), 160–162 (1950). (Russian)

Generalizing the Theorem of Artin, the author shows that if elements $a_1, \cdots, a_n, \cdots$ are associative in sets of three, $[a_i, a_j, a_k]=0$, then the ring which the a's generate will be associative as a subring of an alternative ring containing all the a's. *Marshall Hall* (Columbus, Ohio).

Jacobson, Nathan. Completely reducible Lie algebras of linear transformations. Proc. Amer. Math. Soc. 2, 105–113 (1951).

Let $\mathfrak{L}$ be a linear Lie algebra over a field of characteristic 0. Any x in $\mathfrak{L}$ may be uniquely decomposed into the sum of an element s with simple elementary divisors and a nilpotent element n which commute with each other; if, for any x in $\mathfrak{L}$, s and n still belong to $\mathfrak{L}$, then $\mathfrak{L}$ is called splittable. In particular, the author proves that complete reducibility implies splittability. The main results of the present paper are the following. Let $\mathfrak{L}$ be completely reducible, and $\mathfrak{M}$ a subalgebra of $\mathfrak{L}$. If we assume that $\mathfrak{M}$ has an invariant complementary space in $\mathfrak{L}$ and that its center is splittable, then $\mathfrak{M}$ is completely reducible. If $\mathfrak{M}$ is completely reducible, then its centralizer in $\mathfrak{L}$ is completely reducible. The second one of these results had already been announced (in the case of semi-simple Lie algebras over the complex numbers) by Morozov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 36, 259–261 (1942); these Rev. 4, 187], who gave only a sketch of the proof of a particular case of it. The author's proof is based on a generalization of another result of Morozov [ibid. 36, 83–86 (1942); these Rev. 4, 187] to the effect that every element d of a semi-simple Lie algebra over the complex numbers such that ad d is nilpotent belongs to a simple three-dimensional subalgebra. The proof given by Morozov was incomplete; the author gives a new and correct proof. He establishes that, if $\mathfrak{L}$ is a linear Lie algebra, then, if $\mathfrak{L}$ is completely reducible, every nilpotent element of $\mathfrak{L}$ belongs to a simple 3-dimensional simple algebra, while, conversely, if the latter condition is satisfied and the center of $\mathfrak{L}$ is assumed to be splittable, then $\mathfrak{L}$ is completely reducible. Observing that the multiplication algebra of a Jordan algebra is a Lie triple system, the author derives from his results a criterion of semi-simplicity for Jordan algebras. He also proves that the derivation algebra of a semi-simple Jordan algebra is semi-simple. *C. Chevalley.*

Kolchin, E. R. Picard-Vessiot theory of partial differential fields. Proc. Amer. Math. Soc. 3, 596–603 (1952).

Let F be a partial differential field over an algebraically closed constant field C of characteristic 0, with derivations $\delta_1, \cdots, \delta_m$. A Picard-Vessiot extension of F is defined as an extension G of F such that: (1) the field of constants in G is C, (2) $G=F\langle\eta_1, \cdots, \eta_n\rangle$ with η_i linearly independent over constants and such that $\det(\theta\eta_j)/\det(\theta_0\eta_j) \in F$ for all choices of operators $\theta_1, \cdots, \theta_n$ of form $\theta=\delta_1^{i_1}\cdots\delta_m^{i_m}$ with order $\leqq n$ and for some fixed $\theta_{01}, \cdots, \theta_{0n}$ with $\det(\theta_0\eta_j)\neq 0$. The author develops the Galois theory of such extensions by reducing it to the case of ordinary Picard-Vessiot extensions [Kolchin, Ann. of Math. (2) 49, 1–42 (1948); these Rev. 9, 561], though it is also possible, the author says, to develop it directly. Introduce thus indeterminates $u_1, \cdots, u_m$. $G\langle u_1, \cdots, u_m\rangle$ becomes an ordinary differential field with derivation $D=\sum u_i\delta_i$ and is as such an ordinary Picard-Vessiot extension of $F\langle u_1, \cdots, u_m\rangle$. It is proved in this way that the group of automorphisms of G over F is an algebraic group $\mathfrak{G}$ over C, partial differential fields between G and F are in 1-1 correspondence with algebraic subgroups of $\mathfrak{G}$, and every isomorphism over F of such an intermediate field into G can be extended to an element of $\mathfrak{G}$. Also, Liouvillian extensions are considered, which may be characterized as Picard-Vessiot extensions such that the component of unity of the automorphism group is solvable. *T. Nakayama* (Nagoya).

Theory of Groups

Thurston, H. A. Noncommuting quasigroup congruences. Proc. Amer. Math. Soc. 3, 363–366 (1952).

Dans ce travail on construit un quasigroupe sur lequel il existe deux équivalences compatibles avec la structure de quasigroupe et non commutatives. Ce quasigroupe est en fait le quasigroupe libre engendré par quatre éléments et l'auteur emploie la construction de G. E. Bates et F. Kiokemeister [Bull. Amer. Math. Soc. 54, 1180–1185 (1948); ces Rev. 10, 353]. *J. Riguet* (Paris).

Thurston, H. A. Equivalences and mappings. Proc. London Math. Soc. (3) 2, 175–182 (1952).

Let $A=(x, y, \cdots)$ be a set and $\Sigma=(\sigma, \tau, \cdots)$ a semigroup of mappings of A into itself. When $\mathfrak{q}$ is any equivalence on A, then $\mathfrak{q}^\dagger$ is the equivalence relation on Σ which holds $\sigma\mathfrak{q}^\dagger\tau$ if and only if $(x\sigma)\mathfrak{q}(x\tau)$ for every x in A. Conversely, to every equivalence $\mathfrak{p}$ on Σ there corresponds an equivalence $\mathfrak{p}^\dagger$ on A defined to be the smallest transitive relation containing $\mathfrak{p}^0$ consisting of all pairs $(x\sigma, x\tau)$ for every pair $\sigma\mathfrak{p}\tau$. The correspondences

$$\mathfrak{q}\to\mathfrak{q}^\dagger, \quad \mathfrak{p}\to\mathfrak{p}^\dagger$$

give a one-to-one correspondence between the closed Σ-equivalences on A and the closed congruences on Σ. If $\mathfrak{p}^\dagger=\mathfrak{p}^0$, then $\mathfrak{p}$ is simple. If Σ is a group, then every congruence on it is simple and the closed Σ-equivalences on A all commute. This result is extended to systems where multiplication is defined as a combination of a fixed number of factors. *O. Ore* (New Haven, Conn.).

Ohnishi, Masao. Linear-order on a group. Osaka Math. J. 4, 17–18 (1952).

Verf. gibt einen kurzen Beweis des Kriteriums des Ref. [Math. Z. 52, 483–526 (1949); diese Rev. 11, 497] für die strenge Ordenbarkeit einer Gruppe. *P. Lorenzen.*

Lewis, F. A. The linear congruence group modulo n. Proc. Amer. Math. Soc. 3, 367–368 (1952).

The author states several results on the order and generating systems of the group of linear transformations on m variables whose coefficients are rational integers taken modulo n, in such a way that the determinant of each transformation is prime to n. *T. Szele* (Debrecen).

Kuhn, H. W. Subgroup theorems for groups presented by generators and relations. Ann. of Math. (2) 56, 22–46 (1952).

This paper treats subgroups of free products or of free products with an amalgamated subgroup. The results are too complicated to state in any detail in this review.

The starting point is a demonstration of the equivalence of a problem considered by Reidemeister and Schreier [Schreier, Abh. Math. Sem. Univ. Hamburg 5, 161–183 (1927)] to that of determining free generators for a free group. The author gives a new solution to this problem which can be applied to questions concerning the structure of subgroups of free products or of free products with an amalgamated subgroup. A major result of the paper is a new formulation of the Kurosh subgroup theorem [Math. Ann. 109, 647–660 (1934)] which states that every subgroup of a free product is itself a free product. This new theorem is not only more general than previous formulations but also its proof is more direct. This theory is applied to the commutator groups of free products or of free products with an amalgamated subgroup. *R. M. Thrall.*

Lyndon, R. C. Two notes on nilpotent groups. Proc. Amer. Math. Soc. 3, 579–583 (1952).

Let G_n denote the nth term of the lower central series of a group G, G^m the subgroup generated by the mth powers of the elements of G, and $\{U, V\}$ the group generated by the subgroups U, V of G. As generalization of a theorem due to Rédei and Szép [Monatsh. Math. 55, 200–205 (1951); these Rev. 13, 203] in the first part of this paper the author proves the following theorem. If A and B are subgroups of a nilpotent group G such that $A^{m^s}=1$ for some integer m^s, then, for any $n\geqq 1$, $\{A, B\}_n=\{A^m, B\}_n$ implies $\{A, B\}_n=B_n$. In the second part of the paper, by a uniform method, the following theorems are established. Every finitely generated nilpotent group is definable by a finite set of relations. In a finitely generated nilpotent group the word-problem is decidable. Any nilpotent group possesses a finite basis of identical relations. The latter result answers a question that was raised by B. H. Neumann [Math. Ann. 114, 506–525 (1937)]. *T. Szele* (Debrecen).

Krasner, Marc, et Kaloujnine, Léo. Produit complet des groupes de permutations et problème d'extension de groupes. I. Acta Sci. Math. Szeged 13, 208–230 (1950).

Krasner, Marc, et Kaloujnine, Léo. Produit complet des groupes de permutations et problème d'extension de groupes. II. Acta Sci. Math. Szeged 14, 39–66 (1951).

Krasner, Marc, et Kaloujnine, Léo. Produit complet des groupes de permutations et problème d'extension de groupes. III. Acta Sci. Math. Szeged 14, 69–82 (1951).

Let Γ_i be a group of permutations on a finite set M_i $(i=1, \cdots, s)$. The complete product $\mathfrak{G}_2=\Gamma_1 o \Gamma_2$ is the group of all permutations σ on $M^2=M_1\times M_2$ such that $\sigma(x_1, x_2)=(\sigma_1 x_1, \sigma_2(x_1)x_2)$ where $\sigma_1 \in \Gamma_1$ and $\sigma_2(x_1) \in \Gamma_2$. Then $\mathfrak{G}_i=\Gamma_1 o \Gamma_2 o \cdots o \Gamma_i$ is defined inductively as $\mathfrak{G}_{i-1} o \Gamma_i$ and is called the complete product of $\Gamma_1, \cdots, \Gamma_i$ $(i=1, \cdots, s)$. The complete product is associative but not commutative. If Γ_i has degree d_i and order s_i, then $\Gamma_1 o \Gamma_2$ has degree $d_1 d_2$ and order $s_1 s_2^{d_1}$. A complete product is transitive if and only if each factor is transitive. Let G be a subgroup of $\mathfrak{G}=\mathfrak{G}_s$, let $m=(m_1, \cdots, m_s)$ be a fixed element of $M=M^s$ where $M^i=M_1\times\cdots\times M_i$ $(i=1, \cdots, s)$, and denote by $G_i\langle m\rangle$ the group of all $\sigma \in G$ for which

$$\sigma(m_1, \cdots, m_s)=(m_1, \cdots, m_i, *\cdots*) \quad (i=1, \cdots, s).$$

The set $G=G_0\langle m\rangle, G_1\langle m\rangle, \cdots, G_s\langle m\rangle$ is called the canonical sequence of G associated with m; this is called the canonical sequence of G if m is the identity. Every transitive subgroup G of $\mathfrak{G}$ has a series of subgroups $G=G_0\supset G_1\supset\cdots\supset G_s$ such that (1) G_s contains no invariant subgroup of G and (2) the permutation representation of G_{i-1} given by the cosets of G_i is equivalent to a transitive subgroup $\bar{\Gamma}_i$ of Γ_i. Conversely, any abstract group H which satisfies conditions (1) and (2) is isomorphic to a transitive subgroup of $\mathfrak{G}$. A similar result holds for complete products of abstract groups which are defined as follows: If $\Gamma_1, \cdots, \Gamma_s$ are abstract groups, the complete product $\mathfrak{G}=\Gamma_1 o \Gamma_2 o \cdots o \Gamma_s$ is defined as the complete product of the regular representations of the Γ_i. Clearly $\mathfrak{G}$ is a permutation group on the elements of the cartesian product $\Gamma=\Gamma_1\times\cdots\times\Gamma_s$. The associative law does not hold for complete products of abstract groups. In the third of the papers the theory is applied to the problem of group extensions. *R. M. Thrall* (Ann Arbor, Mich.).

***Kaloujnine, L. Le produit complet des groupes et la théorie d'extension de Schreier.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 203–206. Centre National de la Recherche Scientifique, Paris, 1950.

A summary of papers reviewed above.

Osima, Masaru. On the representations of groups of finite order. Math. J. Okayama Univ. 1, 33–61 (1952).

Let A be an algebra with a unit element 1 over an algebraically closed field K. We may consider A as the representation module of a representation T of the direct product $A\times A'$ of A with its reciprocal algebra A'. If α, β are two elements of A and if $\beta' \in A'$ corresponds to $\beta \in A$, then T represents the element $\alpha\beta'$ of $A\times A'$ by the linear transformation $\xi\rightarrow\beta\xi\alpha$ of A into itself where ξ denotes a variable element of A. It is shown by a direct argument that the multiplicities of the irreducible constituents of T are the Cartan invariants of A. Let B be a subalgebra of A such that $1 \in B$. Every left ideal $\mathfrak{l}$ of B generates a left ideal $\mathfrak{l}^*$ of A. If L is the representation of B belonging to $\mathfrak{l}$, the representation of A belonging to $\mathfrak{l}^*$ is the representation L^* of A induced by L. In particular, the distinct indecomposable components $U_1^{(1)}, U_2^{(1)}, \cdots, U_m^{(1)}$ of the first regular representation of B yield m representations $U_1^{(1)*}, U_2^{(1)*}, \cdots, U_m^{(1)*}$ of A in this manner. Let C be a second subalgebra of A, $1 \in C$. It is shown next that the multiplicities of the irreducible constituents of the restriction of T to $B\times C'$ also appear as the multiplicities of the irreducible representations of C in the restriction of $U_\mu^{(1)*}$ to C. Dually, the same multiplicities appear as the multiplicities of the irreducible representations of B in the restrictions to B of the representations $V_1^{(2)*}, V_2^{(2)*}, \cdots, V_\nu^{(2)*}$ of A induced by the indecomposable components $V_1^{(2)}, V_2^{(2)}, \cdots, V_\nu^{(2)}$ of the second regular representation of C.

These results are used to obtain a new treatment of the basic results of the theory of group representations. In particular, the orthogonality relations are obtained as well for ordinary as for modular group characters. The Frobenius reciprocity theorem for induced characters is generalized as follows. Let φ be an irreducible character of a subgroup H of G and let ψ be an irreducible character of a subgroup J of G. If ψ appears with the multiplicity c in the restriction to J of the character φ^* of G induced by φ, then φ appears with the multiplicity c in the restriction to H of the character ψ^* of G induced by ψ. Nakayama's analogues for the modular case are extended in a similar manner. [It may be remarked that while there is a certain interest in a direct proof, these results can be obtained very easily from the orthogonality relations for group characters.] Further, the multiplication

of characters is studied. Finally, the representations of groups by collineations are treated.

The following new results may be mentioned. If H is a subgroup of G, the ordinary irreducible representations $Z_1, Z_2, \cdots, Z_k$ of G are distributed into maximal classes such that the restrictions to H of representations in different classes do not have a common constituent. These maximal classes are called the H-blocks. Let M be the intersection of H with its conjugates. The number of H-blocks is equal to the number of classes of conjugate elements of G which contain elements of M. The restriction to M of Z_i breaks up into a class of irreducible representations of M associated in G, all appearing with the same multiplicity. It is shown that Z_i and Z_j belong to the same H-block if and only if both belong to the same class of associated representations of M. A similar idea is studied in the modular case.

R. Brauer (Cambridge, Mass.).

Ibrahim, E. M. The plethysm of S-functions. Quart. J. Math., Oxford Ser. (2) **3**, 50–55 (1952).

This paper seeks to make precise a method of D. E. Littlewood for obtaining the reduction of the "plethysm" $\{\lambda\} \otimes \{\mu\}$ of two Schur functions $\{\lambda\}$ and $\{\mu\}$. The language used is that of invariant theory but this is incidental. No reference is made to a paper by the reviewer [Canadian J. Math. **2**, 334–343 (1950); these Rev. **12**, 74] in which the problem is related to the reduction of an induced representation of the corresponding symmetric group and so brought to a systematic solution [cf. also the paper reviewed below]. *G. de B. Robinson* (Toronto, Ont.).

Littlewood, D. E. Modular representations of symmetric groups. Proc. Roy. Soc. London. Ser. A. **209**, 333–353 (1951).

A powerful method of p-graphs and p-quotients is introduced into the modular theory of representations of symmetric groups. It is a refinement of the reviewer's [Jap. J. Math. **17**, 165–184, 411–423 (1941); these Rev. **3**, 195, 196; **4**, 340] hook method and is closely related to Robinson's [Amer. J. Math. **70**, 277–294 (1948); these Rev. **10**, 678] star diagram method. For a partition $(\lambda_1, \cdots, \lambda_t)$, consider $\lambda_1+i-1, \lambda_2+i-2, \cdots, \lambda_t$. Observing that the removal of a p-hook changes this derived set only within mod p and order of terms [reviewer, loc. cit.], the author introduces "p-residue" in place of p-core. The way a partition is related to its p-residue is described by certain S-functions, p in number, whose product is called the "p-quotient", the notion being closely related to the star diagram. Observing that to replace a cycle of order ap by a cycle of order p changes characters only within mod p and applying the orthogonality relation for characters [cf. Chung, Canadian J. Math. **3**, 309–327 (1951); these Rev. **13**, 106; Osima, the paper reviewed below], a simple proof is given for a conjecture of the reviewer, proved formerly by Brauer [Trans. Roy. Soc. Canada. Sect. III.3 **41**, 11–19 (1947); these Rev. **10**, 678] and Robinson [ibid. **41**, 20–25 (1947); these Rev. **10**, 678]; only a half of the conjecture is proved in the present paper (but the author says, in correspondence with the reviewer, that the other half may be treated similarly by his method). The author's [The theory of group characters . . . , Oxford Univ. Press, 1940; these Rev. **2**, 3] theorem on characters of permutations of which the orders of cycles have a common factor is generalized. These are used, with the help of a result of Murnaghan [Amer. J. Math. **59**, 437–488 (1937)], to give one method of decomposing ordinary characters into modular ones. A second method is based on the "p-graph" of a diagram; it is obtained by replacing the node at (i, j) by the least positive integer congruent to $i-j+1$ mod p. Two diagrams have the same p-core if and only if their p-graphs consist exactly of the same integers. The series of restrictions on symmetric subgroups of an irreducible representation give rise to a permutation of the integers composing the p-graph, and the second method is given by its means. (The method of p-graphs has been pushed further forward by Robinson [Proc. Nat. Acad. Sci. U. S. A. **37**, 694–696 (1951); these Rev. **13**, 530].) A method to express the author's [Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41] plethysm product of S-functions is given, which is simpler than, though closely related to, Todd's [Proc. Cambridge Philos. Soc. **45**, 328–334 (1949); these Rev. **10**, 672] and Robinson's [Canadian J. Math. **2**, 334–343 (1950); these Rev. **12**, 74]. *T. Nakayama.*

Osima, Masaru. On some character relations of symmetric groups. Math. J. Okayama Univ. **1**, 63–68 (1952).

Let $\mathfrak{G}$ be a group of finite order. Let k denote the number of classes of conjugate elements and let k^* denote the number of p-regular classes in $\mathfrak{G}$. If $\chi_1, \chi_2, \cdots, \chi_k$ are the irreducible characters of $\mathfrak{G}$, there exist exactly $k-k^*$ linearly independent relations of the form $\sum a_i \chi_i(R) = 0$ which hold for all p-regular elements R of $\mathfrak{G}$ and which have complex coefficients a_i independent of R. A full set of such relations can be obtained at once from the orthogonality relations for group characters. The author uses Murnaghan's recurrence rule [F. D. Murnaghan, Amer. J. Math. **59**, 437–488 (1937)] to obtain a simplified form of these equations in the case when $\mathfrak{G}$ is the symmetric group. *R. Brauer.*

Robinson, G. de B. On a conjecture by J. H. Chung. Canadian J. Math. **4**, 373–380 (1952).

The paper proves, as had been briefly reported earlier [Proc. Nat. Acad. Sci. U. S. A. **37**, 694–696 (1951); these Rev. **13**, 530], a conjecture of Chung [Canadian J. Math. **3**, 309–327 (1951); these Rev. **13**, 106] that the number of ordinary (resp. modular) irreducible representations in a p-block of a symmetric group is independent of its p-core (but is determined by the number of nodes in the core). The ordinary representation part of the conjecture is proved by showing that a diagram with given core, given star diagram and given classes of disjoint constituents of star diagram [Robinson, Amer. J. Math. **70**, 277–294 (1948); these Rev. **10**, 678; Staal, Canadian J. Math. **2**, 79–92 (1950); these Rev. **11**, 415] exists uniquely (which has been observed also by Osima [Nakayama and Osima, Nagoya Math. J. **2**, 111–117 (1951); these Rev. **12**, 672]). The modular representation part is proved by showing that the linear relations between the ordinary irreducible representations modulo p obtained by Chung and Osima by means of the orthogonality and the formula of Murnaghan and the reviewer are complete and that all linear relations between such relations are derived from the ones given by Chung [cf. also the paper reviewed above]. *T. Nakayama* (Nagoya).

Robinson, G. de B. On the modular representations of the symmetric group. II. Proc. Nat. Acad. Sci. U. S. A. **38**, 129–133 (1952).

First the relationship between the p-hook method [cf., e.g., part I, same Proc. **37**, 694–696 (1951); these Rev. **13**, 530] and the p-graph method of Littlewood [the paper reviewed third above] is observed. The ambiguity in Little-

wood's association of a permutation to each indecomposable of the regular representation of a symmetric group, which is "admitted" by the p-graphs of its irreducible components, is shown to be useful in characterizing modular irreducible constituents. The associated permutations lead to a certain "restricting" process, which together with a certain inducing process equivalent to Chung's "minimum complete linkage hypothesis" [Canadian J. Math. 3, 309–327 (1951); these Rev. 13, 106] fits the reciprocity relation of the reviewer [cf. Brauer and Nesbitt, Ann. of Math. (2) 42, 556–590 (1941); these Rev. 2, 309]. *T. Nakayama* (Nagoya).

Robinson, G. de B. On the modular representations of the symmetric group. III. Proc. Nat. Acad. Sci. U. S. A. 38, 424–426 (1952).

Putting $\mathfrak{a}_{ij}=j-i+1$ $(i, j=1, 2, \cdots)$, the author calls the part of $(\mathfrak{a}_{ij})$ overlapped by a Young diagram $[\alpha]$ the graph (α) of $[\alpha]$, in analogy to Littlewood's [the paper reviewed fourth above] p-graph, and correlates a standard diagram s of $[\alpha]$ with a permutation of the numbers in (α), by reading them off by means of the order in s, to reformulate the previous results of Littlewood, loc. cit. and the author [the paper reviewed above] on the modular representations of symmetric groups. The quantity $1/\rho=\mathfrak{a}_{ij}-\mathfrak{a}_{kl}$ is involved in Young's seminormal representation. The author observes the significance of the above permutation that the congruence of the permutations π, π' associated with two standard diagrams implies the congruence of the quantities $1/\rho$, $1/\rho'$ involved in the matrices representing a transposition $(r, r+1)$, in the corresponding seminormal representations, for every r. He hopes that these and the known hook-structure of the diagrams combined will lead to a solution of the problem of modular representations of symmetric groups. *T. Nakayama* (Nagoya).

Murnaghan, F. D. On the multiplication of representations of the linear group. Proc. Nat. Acad. Sci. U. S. A. 38, 738–741 (1952).

The paper describes a method for calculating systematically the products of S-functions. By using a result due to Newell [Proc. London Math. Soc. (2) 53, 356–362 (1951); these Rev. 13, 6] the author improves on an earlier method of his own [Amer. J. Math. 59, 437–488 (1937); 60, 44–65 (1938)].

The method proposed is most easily described by examples. Suppose that it is desired to calculate the product $\{\lambda\}\{21\}$. If $\{\lambda\}$ has one part only, the S-function $\{21\}$ is associated with the master formula

$$\{21\}+\{12\}+\{1^3\}+\{021\}.$$

The product $\{n\}\{21\}$ is obtained by adding n to the first of each partition. If the parts of a partition are not in descending order, the meaning attached to $\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ is taken as $|h_{\lambda_r-s+t}|$ from which definition rules for rearranging the parts may be deduced [see Murnaghan, loc. cit.]. If the partition (λ) has two parts, a longer master formula is needed, and can be found as follows. The partitions beginning with 2 are followed by the one-term master formula for $\{1\}$, to give $\{21\}+\{201\}$. Beginning with 1, this is followed by the one-term master formula for $\{2\}+\{1^2\}$ to give $\{12\}+\{1^3\}+\{102\}+\{1^2\}+\{101^2\}$. Beginning with 0, follow by the one-term master formula for $\{21\}$, to give $\{021\}+\{012\}+\{01^3\}+\{0^221\}$. The whole set of 11 terms gives the master formula. The product $\{\lambda_1, \lambda_2\}\{21\}$ is the sum of 11 S-functions obtained by adding λ_1 to the first part and λ_2 to the second part in the master formula. The method extends to any case. Master formulae are given for partitions of numbers from 1 to 5 for 1 or 2 parts.

D. E. Littlewood (Bangor).

Dynkin, E. B. Topological invariants of linear representations of the unitary group. Doklady Akad. Nauk SSSR (N.S.) 85, 697–699 (1952). (Russian)

A set of topological invariants for equivalence classes of unitary representations of the special unitary group is given. It is stated that analogous results can be obtained for representations of any of the classical groups into another such group. Pontryagin constructed in a Lie group, starting from the multiplication in the group, a notion of product of cycles, in which the homology class of the product depends only on the homology classes of the factors. In the group $U(n)$ of unitary transformations of determinant one in n-dimensional space he set up a set of "primitive" cycles $z_2, \cdots, z_n$ (dimension of z_k is $2k-1$) whose products in this sense generate the homology classes in $U(n)$. The theorem of the present paper is that if φ is a representation of $U(n)$ into $U(N)$, with $z_2, \cdots, z_n$ and $Z_2, \cdots, Z_n$ the respective primitive cycles of these groups, then $\varphi(z_k)$ is homologous in $U(N)$ to $d_k(\varphi)Z_k$, $d_k(\varphi)$ being an integer. If φ and ψ are two such representations, of which one is irreducible, then φ and ψ are equivalent (similar) if and only if

$$d_k(\varphi)/N(\varphi)=d_k(\varphi)/N(\psi) \quad (k=2, \cdots, n).$$

I. E. Segal (Chicago, Ill.).

Dynkin, E. B. Maximal subgroups of the classical groups. Trudy Moskov. Mat. Obšč. 1, 39–166 (1952). (Russian)

The "classical groups" here are taken over the complex field, and so they are the unimodular, orthogonal (determinant 1), and symplectic complex groups. Pertinent earlier references are the author's papers (1) Mat. Sbornik N.S. 30(72), 349–462 (1952); (2) Doklady Akad. Nauk SSSR (N.S.) 75, 333–336 (1950) [these Rev. 13, 904; 12, 589] and (3) Karpelevič, ibid. 76, 775–778 (1951) [these Rev. 12, 585]. In (3) the nonsemisimple maximal subalgebras of semisimple Lie algebras are determined. In (2) there is a brief announcement of the results of the present paper. In (1) all semisimple subalgebras of semisimple Lie algebras are determined; this accomplishes a major part of the task of completing the present investigation by finding the maximal subgroups of the exceptional Lie groups. Paper (1) was written after the present one, although it appeared earlier.

Let G be one of the classical groups, H a maximal subgroup. Case I. H is reducible. Then H consists of all linear transformations in G leaving a certain subspace invariant. Case II. H is irreducible but not simple. Then the vector space is a Kronecker product of two spaces, and H is a direct product of suitable groups on the two factors. This focuses attention on the one remaining case, where H is irreducible and simple. Now it turns out (Theorem 1.5) that every irreducible group of unimodular matrices is maximal in one of the classical groups, with precisely 14 exceptions listed in Table 1. So the problem may be profitably switched to studying all possible inclusion relations among irreducible groups of unimodular matrices. Theorem 2.3 asserts that they are all listed in Table 5.

Chapters I and II are devoted to the statement of all the results and the treatment of Cases I and II. In Chapter III the author develops certain topics in the representation theory of Lie algebras that are needed; these are largely concerned with the structure of the Kronecker product of two representations of a simple Lie algebra. Chapters IV–VI

present the main investigations on the road to Theorems 1.5 and 2.3, by posing and answering the following question: if G is one of the classical groups, in what cases does it happen that a certain irreducible representation of G remains irreducible when restricted to a certain proper subgroup? The three chapters cover the unimodular, symplectic, and orthogonal cases respectively. A lengthy appendix (40 pages) gives a thorough exposition of the representation theory of semisimple Lie algebras. The paper concludes with 30 tables; besides giving the main results, these also record various subsidiary results, and include an expository record of known facts for the reader's convenience.

I. Kaplansky (Chicago, Ill.).

Karpelevič, F. I. Classification of the simple subgroups of the real forms of the group of complex unimodular matrices. Doklady Akad. Nauk SSSR (N.S.) **85**, 1205–1208 (1952). (Russian)

The classification of the simple subalgebras H of a simple Lie algebra G has been carried out in the complex case by Dynkin [Mat. Sbornik N.S. **30(72)**, 349–462 (1952); these Rev. **13**, 904]. Here the author tackles the real case, assuming that G is a real form of the Lie algebra of complex matrices of trace 0. The method is to pass to the complexifications H' and G'. If H' is simple, we regard the problem of classifying H' as solved, and it is in fact equivalent to the theory of representations of a complex simple Lie algebra. Then the embeddings of H, G in H', G' are studied. A central concept is the division of the possibilities for H and G into two types, according as the associated Cartan automorphism is inner or not. For example, it turns out (Theorem 2) that when H and G' are given, H can be contained in at most one real form of G' of the first type, and at most one of the second type. In each case all the possibilities are explicitly determined in Theorems 3–6. Theorem 7 deals with the case where H' is not simple (it is then a direct sum of two isomorphic simple algebras), and in Theorem 8 the theory is extended to the case where H is merely semisimple. There are no proofs.

I. Kaplansky (Chicago, Ill.).

***Tannaka, Tadao. Sôtai genri. [The principle of duality.]** Iwanami Shoten, Tokyo, 1951. v+128 pp. 3.20 Yen.

As is well-known, we owe to the author [Tôhoku Math. J. **45**, 1–12 (1938)] an extension to non-commutative compact groups of Pontrjagin's duality theorem for locally compact abelian groups [cf. M. Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 9–12 (1941); these Rev. **2**, 316]. The present monograph gives a systematic exposition of the duality or reciprocity in modern mathematics. After a brief introduction (§1) together with the explanation (§2) of the fundamental concepts such as group, ring, field, and representation, the dualities in mathematics are expounded in two chapters. §3, which is entitled "the dualities in group theory", includes the theory of normed rings, Haar measure, Plancherel's theorem, Pontrjagin's duality theorem, Tannaka-Krein's duality theorem, and Kawada's duality theorem for group characters. §4 is entitled "the dualities in geometry" and is devoted to the duality theorems of Poincaré and Alexander-Pontrjagin. Reflexivity of Banach spaces and duality for algebraic system are also sketched in §5. A detailed bibliography is given at the end of the book.

K. Yosida (Nagoya).

Kaplan, Samuel. Extensions of the Pontrjagin duality. II. Direct and inverse sequences. Duke Math. J. **17**, 419–435 (1950).

In part I [same J. **15**, 649–658 (1948); these Rev. **10**, 233] the author showed that the class of groups satisfying the Pontrjagin duality is closed under the operation of taking infinite products. In the present paper, the author studies the question whether the class is closed under the operation of taking inverse limits. With this purpose in mind, he first gives a suitable definition of direct limit for groups which need not be discrete. By means of this generalized definition of direct limit, he proves that the limit group of a countable inverse sequence of locally compact groups satisfies the Pontrjagin duality and that its character group is the direct limit of the dual direct sequence of the corresponding character groups. As remarked by the author, the question of whether such a duality theorem holds for general inverse and direct systems remains still unanswered even for locally compact groups. Some of the difficulties involved in attacking this question are discussed at the end of the paper.

In the early part of the paper, the author proves two elementary theorems to the effect that the (full) infinite products of locally compact groups retain two important properties of such groups, namely, the existence of sufficiently many characters and the extendability of characters of subgroups. However, it is still an open question whether or not the weak products also have these properties.

S. T. Hu (New Orleans, La.).

Calabi, Lorenzo. Sur les extensions des groupes topologiques. Ann. Mat. Pura Appl. (4) **32**, 295–370 (1951).

By an extension $E(B, F)$ of a topological group F by B is meant a topological group E containing a closed normal subgroup isomorphic to F, whose quotient group is isomorphic to B. The problem is to determine all possible E's given B and F. It was solved by Schreier for the discrete (i.e., non-topological) case in terms of certain equivalence classes of functions on $B \times B$ with values in F which the author, with Eilenberg-MacLane, calls cocycles (although the analogy is somewhat tenuous unless F is abelian). The present paper discusses the ramifications due to the topology. The extension is said to be fibered if it is possible to map a neighborhood of the identity x_0 in B continuously onto a set of coset representatives in E. If continuity can be achieved only at the identity, the extension is nearly-fibered. The author associates the nearly-fibered (fibered) extensions with the classes of cocycles which are continuous at (in the neighborhood of) the point (x_0, x_0) of $B \times B$. The notion of equivalence of cocycles must be weakened. No discussion appears of extensions which are not at least nearly-fibered, but this is virtually no restriction since E is always nearly-fibered if both B and F are metrizable. After this general study follows a more detailed investigation of certain special classes of extensions concluding with the explicit determination of all Lie groups of dimensions through four. There is a short final section in which the author considers the extension theory for Lie algebras.

A. M. Gleason (Cambridge, Mass.).

Mostow, G. D. On the L^2-space of a Lie group. Amer. J. Math. **74**, 920–928 (1952).

Let G be an n-dimensional analytic group in the sense of Chevalley and let A, B be analytical subgroups of G such that the mapping $(a, b) \to ab$ $(a \in A, b \in B)$ is an analytical isomorphism of $A \times B$ onto G. Let w be a left Haar measure on G, i.e., an alternating n-form on G invariant under left

translations of G, and let ϕ, ψ be left Haar measures of A and B respectively. Denoting by $\phi\times\psi$ the product measure of ϕ and ψ, and normalizing w suitably, the author proves that the scalar function $f(a, b)$ defined by $w=f(\phi\times\psi)$ is the determinant $\Delta(b)$ of a suitable linear representation of the group B and that $f=1$ in case B is either compact or semi-simple, or nilpotent. Then, applying a theorem of Malcev and the reviewer on the topological decomposition of Lie groups, the author obtains the following theorems: 1) an analytical group G is analytically the product of a compact subgroup K and a submanifold E which is analytically isomorphic to a Euclidean space, and a left Haar measure of G is given by the product of a left Haar measure on K and a suitable Euclidean measure on E; 2) a Lie group can be imbedded analytically in a Euclidean space; 3) the L^2 space of a Lie group G has a base of functions analytic on G.

K. Iwasawa (Cambridge, Mass.).

Morita, Kiiti. On group rings over a modular field which possess radicals expressible as principal ideals. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 177–194 (1951).

The author considers rings with a unit element which satisfy the minimum condition for left and right ideals. Such a ring A is said to be quasi-primary if it is indecomposable as a two-sided ideal and if in a representation of A as direct sum of indecomposable left ideals each left ideal appears with the same multiplicity. The ring A is called quasi-primary decomposable if it is isomorphic with a Cartesian sum of quasi-primary rings. It is shown that the radical N of a ring A is both a principal left ideal and a principal right ideal if and only if A is a quasi-primary decomposable and generalized uniserial ring. Let G now be a group of finite order and let A be the group algebra of G over an algebraically closed field of characteristic $p\neq 0$. Then A is quasi-primary decomposable if and only if the irreducible modular representations of each block all have the same degree. Let H denote the maximal normal subgroup of G of an order prime to p and let P be a p-Sylow-subgroup of G. The following conditions are necessary and sufficient for A to be quasi-primary decomposable: The group HP must be normal in G and the factor group G/HP must be abelian. More generally, some results concerning the modular representations of groups G are obtained in which HP is normal and the quotient group G/HP is arbitrary. It is shown that the radical of A is both a principal left ideal and a principal right ideal if and only if HP is normal in G and G/HP is cyclic. Some remarks concerning groups with a cyclic Sylow subgroup P conclude the paper. *R. Brauer.*

Wendel, J. G. On isometric isomorphism of group algebras. Pacific J. Math. **1**, 305–311 (1951).

Let G be a locally compact group with right invariant Haar measure m and group algebra $L(G)$. The latter consists of all (real- or complex-valued) m-integrable functions $x(g)$, $y(g)$ with $xy(g)=\int x(gh)y(h)m(dh)$ and norm $\|x\|=\int|x(h)|m(dh)$. The author uses results of Perlis and Walker [Trans. Amer. Math. Soc. **68**, 420–426 (1950); these Rev. **11**, 638] to give examples of non-isomorphic finite groups G, Γ such that $L(G)$ and $L(\Gamma)$ are isomorphic and homeomorphic. He proves that if T is an isometric isomorphism of $L(G)$ upon $L(\Gamma)$, then there exists an isomorphism τ (topological and algebraic) of G upon Γ, a continuous character χ, and a constant c such that $(Tx)(\tau g)=c\chi(g)x(g)$.

The proof depends on a result of Kawada [Math. Japonicae **1**, 1–5 (1948); these Rev. **10**, 11] who proved that the existence of a positive isomorphism P (i.e., $x(g)\geqq 0$ a.e. implies $(Px)(\gamma)\geqq 0$ a.e. in Γ) of the real group algebra $L(G)$ upon $L(\Gamma)$ implies that G and Γ are isomorphic and homeomorphic. The author shows that such an isomorphism between real group algebras is an isometry and, conversely, every isometry between (real or complex) group algebras gives rise to a positive isomorphism between real subalgebras. (The author has extended his main result in the paper reviewed below.) *M. Henriksen* (Lafayette, Ind.).

Wendel, J. G. Left centralizers and isomorphisms of group algebras. Pacific J. Math. **2**, 251–261 (1952).

Let G be a locally compact group with right invariant Haar measure m and (real or complex) group algebra $L(G)$. The latter consists of all m-integrable (real or complex) functions $x(g)$, $y(g)$ with multiplication given by $(xy)(g)=\int x(gh^{-1})y(h)m(dh)$ and norm $\|x\|=\int|x(h)|m(dh)$. A bounded linear transformation A of $L(G)$ into itself is called a left centralizer if $A(xy)=xAy$ for all x, y in $L(G)$. It is shown that every left centralizer A is of the form $Ax=x^*\mu$, where $(x^*\mu)(g)=\int x(gh^{-1})\,\mu(dh)$, for μ a unique regular measure of bounded variation such that $\|A\|=\operatorname{var}\mu$. Moreover, the left centralizers are closed (in the strong operator topology) in the ring of all bounded operators of $L(G)$. Next it is shown that the norm-preserving left centralizers take the form λR_{g_0}, where λ is a scalar of unit modulus and R_{g_0} is a right translation by some g_0 in G. It is also shown that every left centralizer is a strong limit point of the set of finite linear combinations of right translations. These theorems are used to prove that if T is an isomorphism of $L(G)$ upon the group algebra $L(G')$ of another locally compact group G', which does not increase norm, then there is a topological isomorphism τ of G upon G', a constant c, and a continuous character χ such that $(Tx)(\tau g)=c\chi(g)x(g)$, for all x in $L(G)$ (whence it follows that T is an isometry). This extends an earlier result of the author [see the paper reviewed above]. *M. Henriksen.*

Helson, Henry. On the ideal structure of group algebras. Ark. Mat. **2**, 83–86 (1952).

Soit G un groupe abélien localement compact. Supposons que deux idéaux fermés distincts $I_0\subset I$ de l'algèbre (de convolution) $L^1(G)$ aient le même spectre (soient contenus dans les mêmes idéaux maximaux réguliers). L'auteur démontre alors qu'il existe une infinité d'idéaux fermés intermédiaires. Méthode: 1) En s'appuyant sur un théorème de Kaplansky, on voit que I/I_0 ne saurait avoir la dimension 1. 2) Comme les translations de G opèrent dans I/I_0, et définissent une représentation unitaire de G dans cet espace de Banach, on déduit alors d'un théorème de Godement que cette représentation est réductible, cqfd. La théorie des variétés spectrales de Godement [Ann. Sci. École Norm. Sup. (3) **64**, 119–138 (1947); ces Rev. **9**, 327] permet même de trouver des idéaux intermédiaires, mais ce ne sont probablement pas les seuls.

L. Schwartz (Nancy).

Helson, Henry. Spectral synthesis of bounded functions. Ark. Mat. **1**, 497–502 (1952).

Cet article donne la plus courte démonstration donnée jusqu'ici pour le théorème de Kaplansky [Proc. Nat. Acad. Sci. U. S. A. **35**, 133–136 (1949); ces Rev. **10**, 428; voir aussi Riss, C. R. Acad. Sci. Paris **229**, 12–14 (1949); ces Rev. **11**, 79]. Soit G un groupe abélien localement compact, $\hat{G}$ son dual. Pour une fonction f sur G, $\hat{f}$ sera sa transformée de Fourier. Soit $\phi\in L^\infty(\hat{G})$; son spectre Λ_ϕ dans G est l'ensemble des x tels que $\overline{\langle x, \hat{x}\rangle}$ soit dans le sous-espace vectoriel faible-

ment fermé engendré par les translatées de ϕ. On veut montrer (Kaplansky) que si Λ_ϕ est réduit à l'origine, ϕ est une constante C. Pour celà on considère ϕ comme opérateur de multiplication $\hat{f} \to \phi\hat{f}$ sur $L^2(\hat{G})$; d'où un opérateur $f \to T_\phi f$ dans $L^2(G)$, permutant avec les translations, par transformation de Fourier. On montre d'abord que si $f \in L^2$ est égale à 1 au voisinage d'un point a de G, $T_\phi f$ est égale à une constante C au voisinage de a, C étant indépendante de f et de a. Donc si f est la fonction caractéristique d'un ensemble mesurable relativement compact, $T_\phi f = Cf$; par combinaisons linéaires et passages à la limite, cette relation est vraie pour toute $f \in L^2$, cqfd. Résultat équivalent: si $\hat{f} \in L^1(\hat{G})$ est telle que $f(0)=0$, $\hat{f}$ est limite dans L^1 de $\hat{f}_i$ appartenant à l'idéal (de convolution) engendré par f dans $\hat{G}$, telles que les $\hat{f}_i$ soient nulles dans des voisinages de 0 dans G. On en déduit le théorème démontré dans des cas particuliers par Beurling (cours non publié), Mandelbrojt et Agmon [ibid. **228**, 1394–1396 (1949); ces Rev. **11**, 99], Ditkin [Učenye Zapiski Moskov. Gos. Univ. Matematika **30**, 83–130 (1939); ces Rev. **1**, 336]: Si Λ_ϕ a une frontière ne contenant pas d'ensemble parfait non vide, ϕ est limite faible des combinaisons linéaires des $\overline{(x_i, \hat{x})}$, $x_i \in \Lambda_\phi$. *L. Schwartz* (Nancy).

NUMBER THEORY

***Kaprekar, D. R. Cycles of recurring decimals (from $N=3$ to 161 and some other numbers).** Privately printed, Deolali, 1951. vi+55 pp. 5 rupees.

This work contains tables of the complete periods in the decimal representation of M/N where $0<M<N$, M and N are coprime integers, and $N \leqq 163$ (not 161 as in the title) and prime to 10. Since the period for M/N is a cyclic permutation of that of many other rationals with denominator N, it is possible to save space by listing only those periods which are not cyclic permutations of others. Above each term of a period is written the value of M such that the period for M/N commences with this term. The pamphlet contains also reprinted papers by the author on problems involving the digit of numbers. *D. H. Lehmer.*

***Meister, Fr. Magische Quadrate.** Verlag von Ernst Wurzel, Zürich, 1952. 71 pp.

Certain methods of constructing magic squares are examined in detail and the number of squares constructable by these methods determined for squares of orders 3 to 9. Pandiagonal and bordered squares receive special attention. *R. J. Walker* (Ithaca, N. Y.).

Morgantini, E. Sulla ricerca delle soluzioni intere di un tipo notevole di equazioni diofantee. Rend. Sem. Mat. Univ. Padova **21**, 44–57 (1952).

The author discusses a technique for obtaining classes of solutions in integers of Diophantine equations of the form

$$\sum_{i=1}^{r} a_i x_i^{m_i} = \sum_{j=1}^{s} b_j y_j^{n_j}, \quad m_i, n_j \geqq 1.$$

It depends ultimately upon reducing the equation to the simpler multiplicative type $au^m = bv^n$ by suitable parametrization of x_i and y_j. *R. Bellman* (Santa Monica, Calif.).

Hemer, Ove. On the solvability of the Diophantine equation $ax^2+by^2+cz^2=0$ in imaginary Euclidean quadratic fields. Ark. Mat. **2**, 57–82 (1952).

The equation of the title, with abc square-free, is shown to be solvable non-trivially in the quadratic fields $R(\sqrt{-1})$, $R(\sqrt{-2})$, $R(\sqrt{-3})$, $R(\sqrt{-11})$ if and only if $-bc$, $-ca$, $-ab$ are quadratic residues of a, b, c respectively. This was shown earlier for $R(\sqrt{-1})$ and $R(\sqrt{-3})$ by Th. Skolem [Norsk Mat. Tidsskr. **10**, 50–62 (1928), pp. 50–54]. In the case of the other imaginary Euclidean quadratic field $R(\sqrt{-7})$, another condition is needed for solvability, which stems from the author's method, a reduction method similar to that commonly used for the equation in the case of rational integers. Say that the index of the equation is the middle value of the three norms, $N(ab)$, $N(ac)$, $N(bc)$. Then if the index is sufficiently large, a new equation of lower index can be obtained, and the two equations are solvable or not simultaneously. Thus the problem is reduced to solving certain special cases with small index. The special case $x^2+y^2+z^2=0$ is established as impossible in integers of $R(\sqrt{-7})$, and so are all equations which reduce to this. The author lists all equations of this type having index less than 22. The result mentioned in the first sentence of this review, then, holds for equations in $R(\sqrt{-7})$ except those which reduce to $x^2+y^2+z^2=0$. *I. Niven.*

Stolt, Bengt. On the Diophantine equation $u^2-Dv^2=\pm 4N$. Ark. Mat. **2**, 1–23 (1952).

It has been shown by T. Nagell [Norsk Mat. Tidsskr. **26**, 60–65 (1944); Arch. Math. **2**, 161–165 (1950); Introduction to number theory, Wiley, New York, 1951, pp. 195–212; Arch. Math. **3**, 8–9 (1952); these Rev. **8**, 315; **11**, 714; **13**, 207; **14**, 19] that $u^2-Dv^2=\pm N$, where D is a positive non-square integer, can be solved without the use of either the theory of quadratic forms or the theory of quadratic fields. The author extends this theory to (1) $u^2-Dv^2=\pm 4N$, not simply by replacing N by $4N$, but by relating the equation to $x^2-Dy^2=4$ where Nagell used $x^2-Dy^2=1$. Thus from one solution (u, v) of (1) is obtained an associated solution $u'=\frac{1}{2}(ux+vyD)$, $v'=\frac{1}{2}(uy+vx)$ where (x, y) satisfies $x^2-Dy^2=4$. A complete set of associated solutions forms a class, and each class has a fundamental (minimal) solution. Any equation (1) has but a finite number of classes of solutions, in fact at most 2^n classes in case N is square-free with n prime factors. (The author has carried this part of the theory beyond Nagell's formulation.) The number of classes is given a better bound in many special cases, with exact values being given if N is a prime or the product of two primes. *I. Niven* (Eugene, Ore.).

Nagell, Trygve. Sur un théorème d'Axel Thue. Ark. Mat. **1**, 489–496 (1952).

The theorem of Thue is that for any integer a which is relatively prime to p, there exist positive integers $x<\sqrt{p}$ and $y<\sqrt{p}$ such that $a \equiv \pm x/y \pmod{p}$ for one or the other of the signs. The author proves that at most two such representations exist for any a, and that, except for $p=3, 7, 23, 47$, for every prime there exists at least one number a having two representations, one of each sign. As a corollary the following known result is obtained: the smallest positive odd quadratic non-residue of $p \equiv 7 \pmod 8$ is less than $\sqrt{p}$, except for $p=7, 23$. *I. Niven* (Eugene, Ore.).

Nagell, Trygve. Sur le plus petit non-reste quadratique impair. Ark. Mat. **1**, 573–578 (1952).

A simple method is used to prove that ψ_p, the least positive odd quadratic non-residue of $p \equiv \pm 1 \pmod 8$ is less

than $\sqrt{p}$ except for $p=7$, 23; that $\psi_p \leqq 2+\sqrt{(p-4)}$ for $p\equiv 5 \pmod 8$. It is stated that similar methods will prove that $\psi_p < \sqrt{p}+4$ for $p\equiv 3 \pmod 8$ except $p=131$. Stronger results were obtained for $p\equiv \pm 1 \pmod 8$ by Vinogradov [Trans. Amer. Math. Soc. 29, 218–226 (1927)] by analytic methods, and for $p\equiv \pm 3 \pmod 8$ by A. Brauer [Math. Z. 33, 161–176 (1931); 35, 39–50 (1932)] by elementary methods. Other work has been done by the author [Ark. Mat. 1, 185–193 (1950); these Rev. 11, 640] and by H.-J. Kanold [J. Reine Angew. Math. 187, 169–182 (1950); these Rev. 12, 592]. See also the preceding review. *I. Niven.*

Nagell, Trygve. Sur les restes et les non-restes cubiques. Ark. Mat. 1, 579–586 (1952).

Let $\pi(p, n)$ and $\psi(p, n)$ denote the smallest positive prime n-ic residue and non-residue respectively modulo p. If $p\equiv 1 \pmod 6$, then $\pi(p, 3) \leqq \sqrt{(4p-27)}$ except $p=7$, and better, if p is not of the form $(A^2+27)/4$, then

$$\pi(p, 3) \leqq \sqrt{((4p-1)/27)}.$$

The proof is obtained from a discussion of the representation $p=(A^2+27B^2)/4$, it being established that every divisor of AB is a cubic residue modulo p. Also

$$\psi(p, n) \leqq -1+\sqrt{((p+5)/2)}$$

for $p\geqq 13$, odd $n\geqq 3$, $(n, p-1)>1$, the proof being given in detail in the case $n=3$. Finally it is proved that as $p\to\infty$, $\limsup \psi(p, n)=\infty$, $\limsup \pi(p, n)=\infty$ for $n=2, 3$. Related work on similar questions has been done by L. Rédei [Nieuw Arch. Wiskunde (2) 23, 150–162 (1950); these Rev. 11, 417] and H.-J. Kanold [J. Reine Angew. Math. 188, 74–77 (1950); these Rev. 12, 483]. See also the two preceding reviews. *I. Niven* (Eugene, Ore.).

Delcourte, M. Sur les sommes des résidus quadratiques des nombres premiers. Mathesis 61, 73–79 (1952).

If $p\equiv 1 \pmod 4$, then among the integers $1, 2, \cdots, (p-1)/2$ there are as many quadratic residues as non-residues. However, Dirichlet discovered, in his researches on the class number of quadratic forms, that if $p\equiv -1 \pmod 4$, there are more quadratic residues than non-residues in the same range. In the present paper the sum $\sum_{s=1}^{(p-1)/2}\left(\frac{s}{p}\right)$ is denoted by E_p and is called the quadratic excess of p (although it is not proved that this sum is positive). Formulas are obtained for the sums $\sum_s\left(\frac{s}{p}\right)s$ in terms of E_p for s running from 1 to $p-1$ and, in the case $p\equiv -1 \pmod 4$, for s from 1 to $(p-1)/2$. These are used to derive expressions for the sum of residues and the sum of non-residues among the integers $1, 2, \cdots, p-1$ and also among $1, 2, \cdots, (p-1)/2$. *W. H. Simons* (Vancouver, B. C.).

Borel, Émile. Étude élémentaire de la fréquence des nombres premiers dans les progressions arithmétiques. Rend. Circ. Mat. Palermo (2) 1, 111–128 (1952).

Après avoir rappelé des démonstrations élémentaires de résultats classiques sur les nombres premiers, on montre comment ces résultats peuvent être précisés par certains dénombrements et on donne une démonstration élémentaire du théorème de Dirichlet sur la fréquence des nombres premiers dans une progression arithmétique.

Author's summary.

Fournier, Georges. Sur la distribution moyenne des nombres premiers. C. R. Acad. Sci. Paris 234, 2411–2413 (1952).

Day, J. W. R. A theorem on Bernoulli numbers and the zeta function. J. London Math. Soc. 27, 502–504 (1952).

Proof of the well-known formula $\zeta(2k)=\frac{1}{2}(2\pi)^{2k}B_k/(2k)!$, depending on a recursion formula given by Estermann [same J. 22, 10–13 (1947); these Rev. 9, 234]. *N. G. de Bruijn* (Amsterdam).

Wiebelitz, Rudolf. Über approximative Funktionalgleichungen der Potenzen der Riemannschen Zetafunktion. Math. Nachr. 6, 263–270 (1952).

The author generalizes the Hardy-Littlewood approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$ to $\zeta^k(s)$ where $k>2$ is an integer. Modifying the author's notation, let $\chi(s)=\zeta(s)/\zeta(1-s)$, let $\zeta(\frac{1}{2}+it)=O(t^c)$ where $0<c\leqq 1/6$, let $\beta=3/2$ if $k=3$ but $\beta=k/\{4(1+kc-4c)\}$ if $k>3$, let $a_{\rho,k}$ be the coefficient of $(s-1)^\rho$ in the Laurent expansion of $\zeta^k(s)$ about $s=1$, and let

$$r_{\rho,\nu}=\frac{1}{\nu!}\sum_{\mu=0}^{\rho-\nu-1}(-1)^\mu\binom{k+\mu-1}{\mu}\binom{k-1}{\rho-\mu-\nu-1}.$$

Further, let $d_k(n)$ be the number of representations of n as the product of k positive integers and

$$Q_s(s)=\sum_{n\leqq x}d_k(n)n^{-s}-x^{1-s}\sum_{\nu=0}^{k-1}\log^\nu x\sum_{\rho+1\leqq\beta<\alpha+\beta}a_{-\rho,k}r_{\rho,\nu}(1-s)^{\nu-\rho}.$$

Finally, let $xy=(|t|/2\pi)^k$. The main theorem is that if $x>A>0$ and $y>A$, then

$$\zeta^k(s)=Q_s(s)+\chi^k(s)Q_s(1-s)+O(U_1)+O(U_2)+O(U_3)+O(U_4),$$

where the U's are certain error terms given explicitly. Using the best known value of $c=15/92+\epsilon$ due to Min, the author shows that for $k=3$ the four error terms reduce to $O(x^{1/2-\sigma}|t|^{13/92+\epsilon})$.

Paralleling the proof given by Hardy and Littlewood, the author begins by considering

$$2\pi i\sum_{n\leqq x}d_k(n)n^{-s}\log^{k-1}x/n=(k-1)!\int_{2-\sigma-i\infty}^{2-\sigma+i\infty}\zeta^k(s+z)x^z z^{-k}dz.$$

The path of integration is moved to $\Re(z)=-\gamma$ where $\gamma>0$ is suitably chosen. By evaluating the residues of the integrand at $z=0$ and $z=1-s$ and by a suitable treatment of the new integral along $\Re(z)=-\gamma$, the author obtains the stated result. He then states the mean-value theorem that for $k>2$ and $T>A$:

$$\int_1^T|\zeta(\sigma+it)|^{2k}dt=T\sum_{n=1}^\infty d_k^2(n)\cdot n^{-2\sigma}+O(T^{1/2+k(1-\sigma)/2}\log^{k-1/2}T)+O(T^{k(1-\sigma)}\log^{2k}T).$$

He says that this holds uniformly for $1-k\leqq\sigma\leqq 3/2$ but this is probably a misprint for $1-1/k\leqq\sigma\leqq 3/2$ since the series diverges if $\sigma\leqq 1/2$. *L. Schoenfeld* (Urbana, Ill.).

Dragonette, Leila A. Some asymptotic formulae for the mock theta series of Ramanujan. Trans. Amer. Math. Soc. 72, 474–500 (1952).

The author determines the behaviour of functions of type

$$f(q)=\sum_{n=0}^\infty\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

in the neighbourhood of rational points on the unit circle and obtains a number of results of which the following one is

typical: If $(h, k)=1$, h is odd and $hh' \equiv -1 \pmod{k}$, then for all complex z such that $\Re(z)>0$, $|\Im(z)| \leqq 1/(k+1)$, we have

$$f(e^{\pi i(h+iz)/k}) = \epsilon_{h,k} z^{-1/2} e^{-\pi z/24k + \pi z^{-1}/24k} f(\pm e^{\pi i(h'+ik'z^{-1})/k}) + O(k \log k),$$

where $\epsilon_{h,k}$ is a certain root of unity. These results enable the author to obtain rather precise asymptotic formulae for the coefficients in the power series development

$$f(q) = \sum_{n=0}^{\infty} A(n) q^n.$$

It is proved that

$$A(n) = \sum_{0<k\leqq n^{1/3}} \lambda(k) \exp\{\pi k^{-1} 6^{-1/2}(n-1/24)^{1/2}\} + O(n^{1/2} \log n),$$

where $\lambda(n)$ is a certain exponential sum. This formula constitutes a considerable improvement on a hitherto unproved assertion of Ramanujan. Similar results are stated for a number of related functions. *H. D. Kloosterman.*

Čudakov, N. G., and Rodosskiĭ, K. A. New methods in the theory of Dirichlet's L-functions. Amer. Math. Soc. Translation no. 73, 44 pp. (1952).

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 2(30), 22–56 (1949); these Rev. 13, 824.

Tatuzawa, Tikao. On the zeros of Dirichlet's L-functions. Proc. Japan Acad. 26, no. 9, 1–13 (1950).

The author sharpens the recent results of Čudakov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 31–46 (1948); these Rev. 9, 499] on the finite differences of $\psi(x; k, l)$ so that the "short" arithmetic progressions of Rodosskiĭ are also treated. Let $(k, l)=1$ and $L(s, \chi) \neq 0$ for $\sigma > 1-d/(\log t)^a$ and $t > t_0(k)$. Let $\zeta(\frac{1}{2}+it, w) - w^{-1/2-it} = O(t^c)$; actually, only the following consequence of this assumption is used: $L(\frac{1}{2}+it, \chi) = O(k^{1/2} t^c)$. Also let $x^{(1+4c)/(3+4c)+\epsilon} \leqq y \leqq x$. A reformulation of the author's result is that there exist suitable positive numbers A, B, C depending on ϵ such that

$$\psi(x+y; k, l) - \psi(x; k, l) = \frac{y}{\varphi(k)} - \frac{E(k)}{\varphi(k)} \frac{(x+y)^{\beta_1} - x^{\beta_1}}{\beta_1} + \frac{y}{\varphi(k)} O(R),$$

where $E(k)=1$ if the exceptional real zero β_1 exists and $E(k)=0$ otherwise; also

$$\log R = \begin{cases} -A \log x/\log k & \text{if } C \log k \log\log k \leqq \log x \leqq B(\log k)^{1/a} \\ -A(\log x)^{1-a} & \text{if } B(\log k)^{1/a} \leqq \log x. \end{cases}$$

And, by Siegel's result on the real zeros, the term involving $E(k)$ may be discarded if $k^3 < \log x$.

The principal tool is the result that the number of zeros in $\sigma \leqq \Re(s) \leqq 1$, $|\Im(s)| \leqq T$ of all L-functions with modulus k is

$$O(k^{4(1-\sigma)} T^{4c(1-\sigma)} (T+k)^{2(1-\sigma)} \log^8 kT)$$

if $\frac{1}{2} \leqq \sigma \leqq 1$. This is proved by a method of Ingham. Results related to this are due to Rodosskiĭ [ibid. 12, 47–56 (1948); 13, 315–328 (1949); these Rev. 9, 413; 11, 84] and to Haselgrove [J. London Math. Soc. 26, 273–277 (1951); these Rev. 13, 438].

The author also states without proof the result that each large odd integer N is the sum of three primes p such that $|p - N/3| < N/\log^m N$ for each given m; but this result is weaker than Haselgrove's estimate of N^ϑ for some $\vartheta < 1$. *L. Schoenfeld* (Urbana, Ill.).

Rodosskiĭ, K. A. On the number of zeros of all L-functions with characters of given modulus. Doklady Akad. Nauk SSSR (N.S.) 84, 669–671 (1952). (Russian)

An outline of the proof of the following result is indicated. Let $.9 \leqq \Delta < 1$ and $T \geqq D^{3/2}$. Then for suitable absolute constants D_0 and c the number of zeros $\sigma+it$ of all L-functions, with characters χ modulo D, such that $\Delta \leqq \sigma \leqq 1$ and $|t| \leqq T$ does not exceed $c(DT)^{3(1-\Delta)/(2\Delta)} \log^{12} DT$ provided $D \geqq D_0$. The proof depends on previous results of the author [Ukrain. Mat. Žurnal 3, p. 399 ff. (1951)], not available to the reviewer, and the approximate functional equation for $L(s, \chi)$ in the form due to Čudakov [Ann. of Math. (2) 48, 515–545 (1947); these Rev. 9, 11]. This result should be compared with a related estimate of Tatuzawa in the paper reviewed above which depends on an order estimate for $L(s, \chi)$ on the line $\sigma = 1/2$. *L. Schoenfeld* (Urbana, Ill.).

Ankeny, N. C. Representations of primes by quadratic forms. Amer. J. Math. 74, 913–919 (1952).

Hecke proved that there are infinitely many primes $g = a^2+b^2$, $b^2 = o(g)$. The author proves, by assuming the extended Riemann hypothesis on certain series introduced by Hecke, that there are infinitely many primes $g = a^2+b^2$, $b = O(\log g)$. The proof makes use of several results of A. Selberg on Dirichlet series. *P. Erdős.*

Erdös, P., and Mirsky, L. The distribution of values of the divisor function $d(n)$. Proc. London Math. Soc. (3) 2, 257–271 (1952).

Let $d(n)$ denote the number of divisors of n, and let $D(x)$ denote the number of distinct values of $d(n)$ assumed in the range $1 \leqq n \leqq x$. Let A be the set of integers whose factorisation is of the form $\prod p^{a(p)}$, where p runs through all primes, and $a(p)$ is a non-increasing function of p. An A-number is called a B-number if the sequence of numbers $a(p)+1$ is a sequence of primes (discarding those $a(p)$'s which are zero). $A(x)$ denotes the number of A-numbers not exceeding x; $B(x)$ is defined analogously. It was proved by Hardy and Ramanujan [same Proc. (2) 16, 112–132 (1917)] that

$$\log A(x) \sim 2\pi \cdot 3^{-\frac{1}{2}} \cdot (\log x)^{\frac{1}{2}}/(\log\log x)^{\frac{1}{2}} \quad (x \to \infty).$$

The present authors determine the asymptotic behavior of $\log D(x)$ by showing that

$$\log D(x) \sim \log B(x) \sim 2^{\frac{1}{2}} \cdot (\log A(x))/(\log\log x)^{\frac{1}{2}}.$$

The relation between $D(x)$ and $B(x)$ is studied in detail. It is proved that

$$B(x) \leqq D(x) \leqq B(x) + B(x) O((\log\log x)^2/(\log x)^{\frac{1}{2}}).$$

Some minor problems concerning the distribution of values of $d(n)$ are also studied. *N. G. de Bruijn* (Amsterdam).

Rushforth, J. M. Congruence properties of the partition function and associated functions. Proc. Cambridge Philos. Soc. 48, 402–413 (1952).

The author has made a thorough study of a manuscript left by Ramanujan entitled, "Properties of $p(n)$ and $\tau(n)$ defined by the relations

$$\sum p(n) x^n = \prod_{r=1}^{\infty} (1-x^r)^{-1}, \quad \sum \tau(n) x^n = x \prod_{r=1}^{\infty} (1-x^r)^{24},"$$

excerpts of which have been published by Hardy and Watson. The more fragmentary parts of the work have been filled in by the author who has supplied proofs for a number of new congruence properties of $p(n)$ and $\tau(n)$. Among these

are cited

$$\tau(n)=n^3\sigma_3(n) \pmod{32}.$$

(This is equivalent to a result given by Lahiri [Bull. Calcutta Math. Soc. 39, 33–52 (1947); these Rev. 9, 225]) and

$$p(49n+k)\equiv 0 \pmod{49}, \quad k=19, 33, 40, 47,$$
$$p(121n-s)\equiv 0 \pmod{121}.$$

Results are given for the moduli 13, 17, 19, and 23. These involve the functions $\tau_k(n)$ introduced by Ramanujan and defined by

$$\sum_{n=1}^{\infty}\tau_k(n)=xQ^rR^s\prod_{n=1}^{\infty}(1-x^n)^{24}, \quad 2r+3s=k=2, 3, 4, 5, 7.$$

Here Q and R are essentially the Weierstrass invariants,

$$Q=1+240\sum_{n=1}^{\infty}\sigma_3(n)x^n, \quad R=1-504\sum_{n=1}^{\infty}\sigma_5(n)x^n.$$

An example of the more complicated results obtained for these larger moduli is the identity

$$\prod_{n=1}^{\infty}(1-x^n)\sum_{n=1}^{\infty}p(529n-22)x^n=5\sum_{n=1}^{\infty}\tau_5(n)x^n+23J,$$

where J denotes a power series with integral coefficients. These results are obtained by expanding appropriate powers of the discriminant $Q^3-R^2=12^3\sum\tau(n)x^n$ as polynomials in Q, R, and $P=1-24\sum_{n=1}^{\infty}\sigma(n)x^n$ and reducing the coefficients with respect to the modulus concerned. *D. H. Lehmer.*

Newman, Morris. Remarks on some modular identities. Trans. Amer. Math. Soc. 73, 313–320 (1952).

For primes $p>3$ the author considers the functions (r integral)

$$S_r(\tau)=\sum_{n=0}^{p-1}h^r(W^{-p^n}\tau),$$

where $h(\tau)=\eta(p^2\tau)/\eta(\tau)$; $\eta(\tau)$ is the Dedekind function $\exp(\pi i\tau/12)\cdot\prod_{n=1}^{\infty}(1-x^n)$ and $x=\exp 2\pi i\tau$; W is the matrix $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$, so that the matrices W^{-p^n} ($n=0, 1, \cdots, p-1$) are a set of right representatives for $\Gamma_0(p^2)$ in $\Gamma_0(p)$ (i.e., the group of all modular matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ for which $c\equiv 0 \pmod p$) and therefore S_r is invariant under the substitutions of $\Gamma_0(p)$. The author studies the behaviour of S_r in the parabolic points $i\infty$, 0 of the fundamental region of $\Gamma_0(p)$. The Ramanujan identities for partitions modulo 5, 7 and Mordell's identity for $\tau(n)$ are obtained as special cases. The author derives also some new identities such as

$$\sum_{n=0}^{\infty}p_r(np+\delta)x^n=p_r(\delta)\prod_{n=1}^{\infty}(1-x^n)^r-p^{\frac{1}{2}r-1}x^{\delta}\prod_{n=1}^{\infty}(1-x^{np})^r,$$

where $0<r\leqq 24$, $r(p-1)\equiv 0 \pmod{24}$, $\delta=r(p-1)/24$ and $p_r(n)$ is the generalized partition function defined by $\prod_{n=1}^{\infty}(1-x^n)^r=\sum_{n=0}^{\infty}p_r(n)x^n$. *H. D. Kloosterman.*

Simons, William H. The Fourier coefficients of the modular function $\lambda(\tau)$. Canadian J. Math. 4, 67–80 (1952).

The author extends the method of Hardy-Ramanujan-Rademacher to evaluate the Fourier coefficients of a modular function of level (Stufe) 2, namely $\lambda(\tau)$. *L. K. Hua.*

Grosswald, Emil. On the genus of the fundamental region of some subgroups of the modular group. Amer. J. Math. 74, 86–88 (1952).

Let Γ be the group of modular transformations. Each element corresponds to matrices $M=\pm\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, where a, b, c, and d are integers satisfying $ad-bc=1$. Let p be a prime >3. Let $\Gamma_0(p)$ be the subgroup of Γ satisfying $c\equiv 0 \pmod p$; and $\Gamma_0^0(p)$ be the subgroup of $\Gamma_0(p)$ with $b\equiv 0 \pmod p$ and $\Gamma(p)$ be the subgroup of $\Gamma_0^0(p)$ with $a\equiv d\equiv 1 \pmod p$. The author proves by a uniformizing method that the genera of $\Gamma_0(p)$, $\Gamma_0^0(p)$, and $\Gamma(p)$ are $[p/12]-\frac{1}{2}\{(-1/p)+(-3/p)\}$,

$$[(p+2)(p-1)/12]-1-\tfrac{1}{2}\{(-1/p)+(-3/p)\},$$

and $(p^2-1)(p-6)/24+1$, respectively. The method depends essentially on the existence of a fundamental region corresponding to the generators used, but the author does not explain this fact clearly. *L. K. Hua* (Peking).

Grosswald, Emil. On the parabolic generators of the principal congruence subgroups of the modular group. Amer. J. Math. 74, 435–443 (1952).

Frasch [Math. Ann. 108, 229–252 (1933)] has shown that the congruence modular group $\Gamma(p)$, where p is a prime, is generated by $\tau'=\tau+p$ and $(p-1)p(p+1)/12$ other generators. The author proves that, except $\tau'=\tau+p$, the other generators can be so chosen that none of them is parabolic, for p being a prime satisfying a certain condition. The condition is related to the primitive roots of p. It is conjectured by the author that the condition is always true, but the best known result due to Hua [Bull. Amer. Math. Soc. 48, 726–730 (1942); these Rev. 4, 130] is not enough for the present purpose. *L. K. Hua* (Peking).

Watson, G. L. Sums of eight values of a cubic polynomial. J. London Math. Soc. 27, 217–224 (1952).

The author proves that every positive integer is a sum of eight pyramidal numbers and is a sum of eight values of $(x^3+5x)/6$ with integer $x\geqq 0$. These results are improvements of results due to Hua [Math. Ann. 111, 622–628 (1935); cf. J. Indian Math. Soc. (N.S.) 4, 127–135 (1940); these Rev. 2, 348] who proved similar results for sufficiently large integers. *L. K. Hua* (Peking).

Carlitz, L. Some problems involving primitive roots in a finite field. Proc. Nat. Acad. Sci. U. S. A. 38, 314–318; errata, 618 (1952).

Let $GF(p^{nm})$ be a field of p^{nm} elements. An element γ is said to belong to e if e is the least positive integer such that $\gamma^e=1$; and γ is said to belong to a polynomial $A(x)=\sum_{r=0}^{k}a_rx^r$ if k is the least integer such that $\sum_{r=0}^{k}a_r\gamma^{p^{nr}}=0$. The author proves that the number $N(e, a(x))$ of elements of $GF(p^{nm})$ belonging both to e and $A(x)$ is equal to

$$N(e, a(x))=\phi(e)\Phi(A)p^{-nm}+O(p^{nm(\frac{1}{2}+\epsilon)}),$$

where $\phi(e)$ is the Euler function, while $\Phi(A)$ is the corresponding function for $GF(p^n, x)$. The method depends on the manipulation of character sums. Next he extends a theorem of Davenport on the distribution of primitive roots. Finally he gives an asymptotic formula for the number of solutions of the equation $\alpha=\sum_{i=0}^{s}\gamma_i x_i^{k_i}$. *L. K. Hua.*

Carlitz, L. A theorem of Dickson on irreducible polynomials. Proc. Amer. Math. Soc. 3, 693–700 (1952).

Dickson [Trans. Amer. Math. Soc. 12, 1–18 (1911)] has determined the number of irreducible cubics of the form $Q(x)=x^3-x^2+ax+b^2$, where $a, b \in GF(p^n)$ and $p>2$. In the present paper the author considers the more general problem of irreducibles $Q(x)=x^m+c_1x^{m-1}+\cdots+c_m$ ($c_i \in GF(p^n)$), with preassigned first and last coefficients c_1 and c_m. He first derives an asymptotic formula for the number of such irreducibles, namely: $p^{n(m-2)}/m+O(p^{nm/2})$ ($m\to\infty$). Then he con-

siders irreducibles $Q(x)$ with assigned first coefficient and last coefficient equal to a square (or a non-square) of the field, thus directly generalizing Dickson's problem. Exact results are now obtained. Finally he determines the number of irreducibles with assigned first coefficient. The methods involve certain L-functions for which the Riemann hypothesis is easily established [Davenport and Hasse, J. Reine Angew. Math. 172, 151–182 (1934)].

A. L. Whiteman (Princeton, N. J.).

Skolem, Th. On a certain connection between the discriminant of a polynomial and the number of its irreducible factors mod p. Norsk Mat. Tidsskr. 34, 81–85 (1952).

Let $f(x)$ be a polynomial of degree n with rational integral coefficients and discriminant D. If p is a prime, $p|D$, and $f(x)\equiv f_1(x)\cdots f_s(x) \pmod p$, where the $f_i(x)$ are irreducible $(\bmod p)$, it is proved that $(D/p)=(-1)^{n-s}$, where (D/p) is the Legendre symbol. (This theorem is due to Stickelberger [Verh. 1. Internat. Math.-Kongress. Zürich, 1897, Teubner, Leipzig, 1898, pp. 182–193]; see also Voronoĭ [Verh. 3. Internat. Math.-Kongress. Heidelberg, 1904, Teubner, Leipzig, 1905, pp. 186–189].)

L. Carlitz.

Specht, Wilhelm. Zur Zahlentheorie der Polynome. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 139–146 (1952).

Let $\mathfrak{P}_n(x)$ denote the set of polynomials

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n,$$

where the a_i are rational integers such that $|a_i|\leqq x$, $i=1,\cdots,n$; let $I_n(x)$ denote the number of irreducible polynomials in $\mathfrak{P}_n(x)$. The following results are proved:

$$I_2(x)=4x^2-2x\log x+O(x),$$
$$I_n(x)=2^nx^n+O(x^{n-1}) \qquad (n>2).$$

[Cf. Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 2, Springer, Berlin, 1925, p. 161, no. 266.]

L. Carlitz (Durham, N. C.).

Specht, Wilhelm. Zur Zahlentheorie der Polynome. II. Math. Nachr. 7, 105–126 (1952).

Let $\mathfrak{R}_n(x)$ denote the set of polynomials

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n$$

with rational integral coefficients such that

$$a_1^2+a_2^2+\cdots+a_n^2\leqq x^2;$$

let $I_n(x)$ denote the number of irreducible polynomials in $\mathfrak{R}_n(x)$ and let $R_n(x)$ denote the number of reducible polynomials in $\mathfrak{R}_n(x)$. Among the results of the paper the following may be quoted:

$$R_2(x)=2x\log x+4\beta_1x+O(x^{\frac12}),$$
$$R_3(x)=\alpha_2\beta_2x^2-\tfrac14x\log^2x+O(x\log x),$$
$$R_4(x)=\alpha_2\beta_2x^2+O(x^2\log x),$$
$$R_n(x)=\alpha_{n-1}\beta_{n-1}x^{n-1}+O(x^{n-\frac12}) \qquad (n\geqq5);$$
$$I_n(x)=\alpha^nx^n-\alpha_{n-1}\beta_{n-1}x^{n-1}+O(x^{n-\frac12}) \qquad (n\geqq5).$$

Here α_n, β_n denote constants independent of x.

L. Carlitz (Durham, N. C.).

Specht, Wilhelm. Zur Zahlentheorie der Polynome. III. Math. Nachr. 7, 127–150 (1952).

In the notation of the preceding review, the following results are proved:

$$R_3(x)=\alpha_2\beta_2x^2-\tfrac14x\log^2x-\tfrac14(1+6\beta_1)x\log x+O(x),$$
$$R_4(x)=\alpha_2\beta_2x^2+\alpha_2\beta_2x^2\log x+O(x^2).$$

L. Carlitz (Durham, N. C.).

Inaba, Eizi. Note on relatively complete fields. Nat. Sci. Rep. Ochanomizu Univ. 3, 5–9 (1952).

Suppose that the field k admits the valuation V of rank 1. The author re-examines several results on the relative completeness of k with respect to V, which were originally due to Ostrowski and the reviewer. For example, it is shown that the validity of Hensel's Lemma for primitive polynomials in k (relative completeness) is equivalent to the existence of unique prolongations of V to finite extensions of k, or also that every separable polynomial without root in k has a neighborhood such that every polynomial in the latter is without roots in k. Results of this type finally imply that units sufficiently close to 1 have nth roots in a relatively complete field provided n is relatively prime to the characteristic of k.

O. F. G. Schilling (Chicago, Ill.).

Cohn, Harvey. Note on fields of small discriminant. Proc. Amer. Math. Soc. 3, 713–714 (1952).

If $n=\frac12(p-1)$, where p is a prime greater than 3, it is well known that there is a totally real field of degree n and discriminant p^{n-1}, namely the field generated by $\cos 2\pi/p$. The author observes that this is not always the totally real field of degree n of least discriminant. Let $l>5$ be a prime. Then, firstly, the field generated by $\cos 2\pi/l$ and $\sqrt5$ is a totally real field of degree $l-1$ and discriminant $l^{l-3}5^{(l-1)/2}$. Secondly, the field generated by $\{3(2-2\cos 2\pi/l)\}^{1/2}$ is a totally real field of degree $l-1$ and discriminant $l^{l-3}3^{(l-1)/2}$. Thus, for example, the first construction gives a totally real field of degree 6 whose discriminant is $7^4\times5^3$.

H. Davenport.

Lewis, D. J. Cubic homogeneous polynomials over p-adic number fields. Ann. of Math. (2) 56, 473–478 (1952).

Let K be a complete discretely-valued field with finite residue field. For each integer $r\geqq1$ there exists a smallest integer $\varphi(r)>0$ such that every form over K in n arguments and of degree r has a nontrivial zero in K if $n\geqq\varphi(r)$ [R. Brauer, Bull. Amer. Math. Soc. 51, 749–755 (1945); these Rev. 7, 108]; it is known that $\varphi(r)>r^2$, and [Hasse, J. Reine Angew. Math. 153, 113–130 (1923)] that $\varphi(2)=5$. Dem'yanov showed [Doklady Akad. Nauk SSSR (N.S.) 74, 889–891 (1950); these Rev. 12, 315] that $\varphi(3)=10$ provided the residue field characteristic $p\neq3$. The present paper contains a different proof (arrived at independently and at the same time as Dem'yanov's) that $\varphi(3)=10$, valid even if $p=3$.

E. R. Kolchin (New York, N. Y.).

Ankeny, N. C., Artin, E., and Chowla, S. The class-number of real quadratic number fields. Ann. of Math. (2) 56, 479–493 (1952).

Let F be a real quadratic number field with discriminant d, and let h denote its class number. Write the fundamental unit $\epsilon>1$ of F in the form $\epsilon=\frac12(t+u\sqrt d)$. For any real number w, let $[w]$ denote the largest integer $\leqq w$. Let x denote the quadratic character $(\bmod d)$ which is defined by: $x(n)=(d/n)$, where (d/n) is the Kronecker (extended) quadratic residue symbol. The following results are proved. Theorem 1. Suppose that $d=pm$, where p is an odd prime, and m an integer >1. Then

$$-2\frac{h}{t}-c\equiv\sum_{0<n<d}\frac{x(n)}{m\cdot n}\left[\frac{n}{p}\right] \pmod p,$$

where $c=1$ if $p>3$, and $c=m+1$ if $p=3$. Theorem 2. Suppose that $d=3q$, where q is square-free and $q\equiv1 \pmod 3$. Then $-uh/t\equiv H \pmod 3$, where H is the class number of the imaginary quadratic field generated by $\sqrt{-q}$. Theorem

3. If d is an odd prime p, then

$$4\frac{u}{t}h \equiv -\sum_{1\le n<p} \frac{1}{gn}\left(\frac{n}{p}\right)\left[\frac{gn}{p}\right] \pmod{p},$$

where g is any quadratic non-residue (mod p). If $p\equiv 5 \pmod 8$, then

$$4\frac{u}{t}h \equiv -\sum_{1\le n<p/4} \frac{1}{n}\left(\frac{n}{p}\right) \pmod{p}.$$

Theorem 4. Suppose that $d=pm$, where p is a prime $>\sqrt{d}$, and let the numbers C_n be defined by the identity:

$$\sum_{n=1}^{\infty}\frac{C_n t^n}{n!}=\frac{1}{e^{mt}-1}\sum_{k=1}^{m}X(k)e^{kt},$$

where X is the unique real character (mod m) which satisfies: $x(n)=(n/p)X(n)$. Then $-2hu/t\equiv C_{\frac{1}{4}(p-3)} \pmod p$.

G. Hochschild (Urbana, Ill.).

Lang, Serge, and Tate, John. On Chevalley's proof of Luroth's theorem. Proc. Amer. Math. Soc. **3**, 621–624 (1952).

Let k be a function field in one variable over a constant field k_0 and let g be its genus. It is shown first that if k is separably generated over k_0, then the genus of any function subfield of k containing k_0 (necessarily separably generated) is at most equal to g. This generalizes an argument of Chevalley [Introduction to the theory of algebraic functions of one variable, Amer. Math. Soc., New York, 1951; these Rev. **13**, 64] in proving Lüroth's theorem. On the other hand, if k is not separably generated over k_0 and $g=0$ (k being of characteristic 2 necessarily), then k has hyperelliptic subfields containing k_0 of arbitrarily high genus; the proof uses the normal form of fields of genus 0 [cf. Artin, Algebraic numbers and algebraic functions, v. I, New York Univ., New York, 1951; these Rev. **13**, 628] and the arithmetic of inseparable quadratic extensions of a rational field.

T. Nakayama (Nagoya).

Tate, John. The higher dimensional cohomology groups of class field theory. Ann. of Math. (2) **56**, 294–297 (1952).

It is shown that the Galois cohomology group $H^r(G, A)$ in the idèle-class group A of an algebraic number field (or the multiplicative group of a p-adic number field) is canonically isomorphic to $H^{r-2}(G, Z)$ where Z is the additive group of rational integers ($r>2$). Let first G be a finite group and A any abelian G-group. Let α be a 2-cohomology class of G in A, and $\bar{A}$ be the Artin splitting group for α. It is proved that the following two axioms are equivalent: (1) $H^1(U, A)=0$, and $H^2(U, A)$ is cyclic of the same order as U, generated by the restriction of α to U, for all subgroups U of G; (2) $H^1(U, \bar{A})=H^2(U, \bar{A})=0$ for all subgroups U of G. The proof depends on two exact sequences $\bar{A}/A\simeq I, R/I\simeq Z$, where R is the group ring of G over Z, considered as a G-module, and I is its ideal generated by the elements $\sigma-1$ ($\sigma \in G$). These exact sequences entail exact sequences for 0-, 1-, 2-cohomology groups in I, A, $\bar{A}$ and in Z, I, R, respectively. They together lead to the theorem. Combined with the fact that (2) implies the vanishing of higher cohomology groups, due to Serre, Lyndon, Hochschild and the reviewer [cf. Hochschild and Nakayama, Ann. of Math. (2) **55**, 348–366 (1952); these Rev. **13**, 916], it leads to the fact that $H^r(G, A)$ is isomorphic with $H^{r-2}(G, Z)$; the isomorphism is given by cup product with α. As the fundamental (or canonical) Galois 2-cohomology class α in the idèle-class group satisfies (1) [Hochschild and Nakayama, loc. cit.], the statement at the opening of the review follows. The vanishing of $H^3(G, A)$ in this class field theory case, proved independently by Hochschild, is a particular instance. The author promises a subsequent paper in which negative-dimensional cohomology groups will be introduced and the reciprocity law will be regarded as another special case of the result.

T. Nakayama (Nagoya).

Chalk, J. H. H. The minimum of a non-homogeneous bilinear form. Quart. J. Math., Oxford Ser. (2) **3**, 119–129 (1952).

Consider values of the real polynomial

$$B=(\alpha x+\beta y+c_1)(\gamma z+\delta t+c_2),$$

where $\Delta=\alpha\delta-\beta\gamma\neq 0$, for integral x, y, z, t such that $xt-yz=\pm 1$. Then there is a solution of $|B|\le\frac{1}{4}|\Delta|$. Furthermore, if α/β, γ/δ are irrational and $(c_1, c_2)\neq(0, 0)$, then $|B|\le\frac{1}{8}|\Delta|$. The case $(c_1, c_2)=(0, 0)$ has been treated by Davenport and Heilbronn [same J. **18**, 107–121 (1947); these Rev. **9**, 79]. The critical lattices of $|(x+c)y|\le 1$ for $c\neq 0$ have determinant $2+\sqrt{2}$. This is a consequence of the following lemma: Let $L_1=\alpha x+\beta y$, $L_2=\gamma x+\delta y$, $\Delta=\alpha\delta-\beta\gamma\neq 0$, and $c\neq 0$. Then there exist relatively prime integers x, y such that $|(L_1+c)L_2|\le k|\Delta|$ where $k=1-1/\sqrt{2}$; here k is least possible. If α/β is rational, then $|(L_1+c)L_2|<m|\Delta|$ where $m=1/4$.

L. Tornheim.

Davenport, H. Note on a result of Chalk. Quart. J. Math., Oxford Ser. (2) **3**, 130–138 (1952).

In the notation of the paper above it is shown that $m=1/4$ is least possible by exhibiting suitable L_1, L_2. If the condition that x, y be relatively prime is replaced by $(x, y)\neq(0, 0)$, then m can be replaced by 1/4.1; also an example is given to prove that the least possible value of m exceeds 1/5.06.

L. Tornheim (Ann Arbor, Mich.).

Kanagasabapathy, P. On the product of two linear forms, one homogeneous and one non-homogeneous. Quart. J. Math., Oxford Ser. (2) **3**, 197–205 (1952).

The constants 4.1 and 5.06 in the preceding review are improved to 4.2575 and 4.28471, respectively.

L. Tornheim (Ann Arbor, Mich.).

Ramanathan, K. G. Abelian quadratic forms. Canadian J. Math. **4**, 352–368 (1952).

The generalized symplectic (g.s.) group $G(n, K)$ over a field K is the set of all $2n\times 2n$ matrices $\mathfrak{M}$ with elements in K and such that $\mathfrak{M}'\mathfrak{J}\mathfrak{M}=k\mathfrak{J}$, where $\mathfrak{J}$ is the matrix $\left(\begin{smallmatrix}\mathfrak{O} & \mathfrak{E}\\ -\mathfrak{E} & \mathfrak{O}\end{smallmatrix}\right)$ ($\mathfrak{O}$ is the zero matrix and $\mathfrak{E}$ the unit matrix of order n; $\mathfrak{M}'$ is the transpose of $\mathfrak{M}$) and $k\neq 0$ is an element of K. It is called the kernel of $\mathfrak{M}$. If $k=1$, the matrix $\mathfrak{M}$ is symplectic (s.). A g.s. matrix $\mathfrak{S}$ is called abelian if it is symmetric. The quadratic form $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ is called abelian if its matrix $\mathfrak{S}$ is an abelian g.s. matrix. Then (A) $\mathfrak{S}=\left(\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{B} & \mathfrak{C}\end{smallmatrix}\right)$, where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are $n\times n$ matrices such that $\mathfrak{A}=\mathfrak{A}'$, $\mathfrak{B}=\mathfrak{B}'$, $\mathfrak{A}\mathfrak{B}'=\mathfrak{B}\mathfrak{A}$, $\mathfrak{A}\mathfrak{C}-\mathfrak{B}^2=k\mathfrak{E}$. There exists a s. matrix $\mathfrak{P}$ such that $\mathfrak{P}'\mathfrak{S}\mathfrak{P}=\left(\begin{smallmatrix}\mathfrak{E} & \mathfrak{O}\\ \mathfrak{O} & k\mathfrak{D}^{-1}\end{smallmatrix}\right)$, where $\mathfrak{D}$ is a diagonal matrix with elements in K. The notions of abelian hermitian form and abelian hermitian matrix (over a field $K_0=K(\sqrt{d})$ which is a quadratic extension of a field K) can be defined accordingly. The determinant $|\mathfrak{S}|$ of an abelian (or hermitian) matrix $\mathfrak{S}$ is k^n. Two integral g.s. matrices $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are called associated if an integral s. (modular) matrix $\mathfrak{P}$ exists such that $\mathfrak{M}_1=\mathfrak{M}_2\mathfrak{P}$. The number of classes of associate matrices with a given kernel is shown to be finite. The author develops a reduction theory for abelian forms. The analogy of the results with those of the theory of binary quadratic forms is strik-

ing: Two abelian matrices $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are called s. equivalent if a modular matrix $\mathfrak{P}$ exists such that $\mathfrak{S}_2=\mathfrak{P}'\mathfrak{S}_1\mathfrak{P}$. A positive definite abelian form (A) is called reduced if (i) the determinant $|\mathfrak{A}|$ is minimal; (ii) $\mathfrak{A}$ is reduced in the sense of Minkowski; (iii) the elements of $\mathfrak{B}^{-1}\mathfrak{A}$ are in absolute value $\leqq\frac{1}{2}$. Then (generalization of the Gauss-Minkowski inequality): $|\mathfrak{A}|\cdot|\mathfrak{C}|\leqq a_1\cdots a_n c_1\cdots c_n\leqq\mu_n|\mathfrak{S}|$, where the a_i and c_i are the diagonal elements of the matrices $\mathfrak{A}$ and $\mathfrak{C}$ respectively and μ_n depends on n only. There is only a finite number of positive reduced forms with a given kernel. The author gives also a short discussion of the reduction of indefinite abelian forms and of abelian hermitian forms, and mentions some further generalizations.

H. D. Kloosterman (Leiden).

Sawyer, D. B. The number of non-homogeneous lattice points in n-dimensional point sets. Proc. Cambridge Philos. Soc. **48**, 735–736 (1952).

Let R be a set in Euclidean n-space and let $\Delta'(R)$ be the g.l.b. of determinants of non-homogeneous lattices with no point in R. Suppose $m\geqq 2$ and integral. If Λ is a non-homogeneous lattice of non-zero determinant $d(\Lambda)$ such that $\Delta'(R)>md(\Lambda)$, then R contains at least $m+n-1$ points of Λ. An example is presented to show that for $m=2$ this result is best possible. The proof, of a geometrical nature, is simple.

L. Tornheim (Ann Arbor, Mich.).

Sawyer, D. B. The number of non-homogeneous lattice points in n-dimensional point sets. Proc. Cambridge Philos. Soc. **49**, 156–157 (1953).

Same as the paper reviewed above.

Rogers, C. A. The reduction of star sets. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 59–93 (1952).

A set S of points in Euclidean n-space is defined to be a star set if $X\in S$ implies $\lambda X\in S$, $-1\leqq\lambda\leqq 1$. If, in addition to this, the points λX, $-1<\lambda<1$, are interior points of S, then S is defined to be a star body. $\Delta_M(S)$ is defined to be the lower bound of the determinants of the lattices whose only point interior to a star body S is the origin. A star body S is defined to be reducible in the sense of Mahler if it contains a star body T as a proper subset with $\Delta_M(S)=\Delta_M(T)$. The present paper starts from the following problem of Mahler [Nederl. Akad. Wetensch., Proc. **49**, 444–454 (1946); these Rev. **8**, 12]. Does every bounded star body S contain an irreducible star body T with $\Delta_M(S)=\Delta_M(T)$? In answer to this problem a plane, bounded, reducible star body S is constructed which contains no irreducible star body T with $\Delta_M(S)=\Delta_M(T)$.

The author introduces the following modifications of the basic concepts. A lattice is defined to be admissible for a star set S if it contains no points of S except the origin. $\Delta(S)$ is the lower bound of the determinants of admissible lattices. A critical lattice is a limit lattice of admissible lattices with determinant $\Delta(S)$; $\Delta_M(S)\leqq\Delta(S)$. Examples of star sets, one bounded by straight lines, are given for which $\Delta_M(S)<\Delta(S)$. The principal result of the paper is that every bounded star set S, with $\Delta(S)>0$, contains an irreducible star set T with $\Delta(T)=\Delta(S)$. In other words the analogue of Mahler's problem with the modified concepts has an affirmative answer.

In the course of the proof a point X of S is defined to be primitively irreducible if X is a point of a lattice Λ with $d(\Lambda)<\Delta(S)$ and such that the only lattice points of S are of the form mX, $m=0, \pm1, \pm2, \cdots$. A point X of S is defined to be an outer boundary point of S if λX is not in S for $\lambda>1$. The author shows first that a star set is irreducible if every outer boundary point is primitively irreducible or the limit point of such points. The final irreducible star subset T of S is obtained by progressively eliminating, in a finite number of steps, those outer boundary points of S which are not irreducible. Results analogous to those of Mahler are obtained. Among these is that every outer boundary point of an irreducible star set S with $\Delta(S)<\infty$ is a point of a critical lattice of S.

In a second section of the paper the author considers bounded star sets S whose boundary points $(x_1, x_2, \cdots, x_n)$ satisfy an equation

$$G(x_1, x_2, \cdots, x_n)=0$$

where $G(x_1, x_2, \cdots, x_n)$ is a polynomial. By methods analogous to those of the previous section he shows that S contains an irreducible star set T with $\Delta(T)=\Delta(S)$ where the boundary points of T also satisfy a polynomial equation.

D. Derry (Vancouver, B. C.).

Ap Simon, H. A method of finding the critical lattices of spheres containing the origin. Quart. J. Math., Oxford Ser. (2) **3**, 91–93 (1952).

Let K be a closed three-dimensional star body which is the union of a strictly convex body K_1 containing the origin as an inner point and its reflection K_2 in the origin. A grid G is the set of 27 points $aA+bB+cC$, where A, B, C are independent points and each of a, b, c takes the values -1, 0, 1. The determinant of G is the determinant of the lattice generated by A, B, C. A grid G is called admissible for K if A, B, C lie on K and no grid point except 0 is an inner point of K. In the set of admissible grids a grid is critical if its determinant is minimal. The problem of determining critical lattices is transformed to that of finding critical grids by a result that the equivalent lattice of a critical grid of K is critical if it is admissible. The author outlines how this result can be applied to the case of K_1 a sphere to find the critical lattices of K and refers to his thesis for details.

L. Tornheim (Ann Arbor, Mich.).

Drazin, M. P., and Griffith, J. Stanley. On the decimal representation of integers. Proc. Cambridge Philos. Soc. **48**, 555–565 (1952).

Let $r\geqq 2$. Put $n=\sum_{k=0}^{\infty}\alpha_k(r, n)r^k$, $0\leqq\alpha_k(r, n)<r$, where n, k, r, $\alpha_k(r, n)$ are integers. Put

$$a_t(r, n)=\sum_{k=0}^{\infty}\alpha_k{}^t(r, n), \quad A_t(r, n)=\sum_{m=1}^{n-1}a_t(r, m).$$

The authors investigate asymptotic properties of $A_t(r, n)$, sharpening previous results of Bush, Mirsky, and Bellman and Shapiro. Here are some of their results: Put

$$\delta_t(r)=r^{-1}\sum_{p=1}^{r-1}p^t, \quad F_t(r, n)=\delta_t(r)\frac{n\log n}{\log r},$$

$$\Delta_t(r, n)=\frac{F_t(r, n)-A_t(r, n)}{n\delta_t(r)}.$$

Mirsky proved that $\Delta_1(r, n)$ is a bounded function of n. The authors prove that, for all r, t, and n, $\Delta_t(r, n)\geqq 0$, $\Delta_t(2, n)<1$, and for $r>3$

$$\Delta_t(r, n)<\frac{r-1}{r-2}\frac{\log(r-1)}{\log r}.$$

The authors also discuss to what extent their results are best possible.

P. Erdős (Los Angeles, Calif.).

Sandham, H. F. The approximation of radicals by rational means. Amer. Math. Monthly **59**, 622–624 (1952).

ANALYSIS

Mohr, Ernst. Beitrag zur Theorie der konvexen Funktionen. Math. Nachr. 8, 133–148 (1952).

This paper is concerned with discontinuous convex functions. Let f be a real-valued function defined on a convex domain $\mathfrak{D}$ in a k-dimensional Euclidean space E_k. Then f is said to be convex if $f(\frac{1}{2}(X+X'))\leqq\frac{1}{2}\{f(X)+f(X')\}$ for each pair of points X, X' in $\mathfrak{D}$. It is proved that if φ is a function defined and convex on the set $\mathfrak{R}$ of all rational points in $\mathfrak{D}$, then φ can be extended to a function $\bar{\varphi}$ convex and continuous in $\mathfrak{D}$. This result is then applied in two directions.

(1) A theorem of Bernstein and Doetsch [Math. Ann. 76, 514–526 (1915)] is generalized to k-dimensional Euclidean space as follows. Let f be convex in $\mathfrak{D}$. If f is not continuous in $\mathfrak{D}$, then its graph is everywhere dense either in the whole $(k+1)$-dimensional strip determined by $\mathfrak{D}$ or in the part of this strip above a continuous convex surface.

(2) A one-to-one correspondence is obtained between convex functions on $\mathfrak{D}$ and continuous convex functions on a space of infinite dimension. Let $\mathfrak{H}$ (with elements α, etc.) be a Hamel basis (over the rational field) for the set E_1 of all real numbers. A space $E_{\mathfrak{H}}$ is defined, the points of which are all real-valued functions $\rho(\alpha)=\rho_\alpha$ defined on $\mathfrak{H}$ such that the 'coordinates' ρ_α are zero except at a finite number of elements α of $\mathfrak{H}$. If $\alpha_1, \cdots, \alpha_n \in \mathfrak{H}$, the 'coordinate space' $\mathfrak{R}=\mathfrak{R}(\alpha_1, \cdots, \alpha_n)$ is the set of all points P in $E_{\mathfrak{H}}$ such that ρ_α is zero if α is not one of the elements $\alpha_1, \cdots, \alpha_n$. The space $E_{\mathfrak{H}}$ is topologized by calling a subset open if its intersection with each coordinate space $\mathfrak{R}$ (regarded as a Euclidean space) is open. $\mathfrak{R}_{\mathfrak{H}}$ is the set of all P in $E_{\mathfrak{H}}$ with rational coefficients and with its topology induced from $E_{\mathfrak{H}}$.

Since $\mathfrak{H}$ is a Hamel basis for E_1, there is a one-to-one correspondence $E_1 \leftrightarrow \mathfrak{R}_{\mathfrak{H}}$ given by $x=\sum r_\alpha \alpha$. Let this Hamel mapping be denoted by $x \leftrightarrow R$. If $\mathfrak{D}$ is an open interval (a, b) in E_1, the set $\bar{\mathfrak{D}}_{\mathfrak{H}}=[P \in E_{\mathfrak{H}} \mid a<\sum \rho_\alpha \alpha<b]$ is an open convex set in $E_{\mathfrak{H}}$ and the set $\mathfrak{D}_{\mathfrak{H}}=\bar{\mathfrak{D}}_{\mathfrak{H}} \cap \mathfrak{R}_{\mathfrak{H}}$ is an open rationally-convex set in $\mathfrak{R}_{\mathfrak{H}}$ corresponding to $\mathfrak{D}$ under the Hamel mapping. If f is convex on $\mathfrak{D}$, then the function Φ given by $\Phi(R)=f(x)$ is convex on $\mathfrak{D}_{\mathfrak{H}}$. Extending Φ on each coordinate space, a function $\bar{\Phi}$ is obtained which is convex and continuous on $\bar{\mathfrak{D}}_{\mathfrak{H}}$. *F. F. Bonsall* (Newcastle-upon-Tyne).

Hyers, D. H., and Ulam, S. M. Approximately convex functions. Proc. Amer. Math. Soc. 3, 821–828 (1952).

Une fonction réelle f définie sur un ensemble convexe E de l'espace euclidien R^n est dite ϵ-convexe si, quels que soient les points x, y de E, $f(\alpha x+\beta y)\leqq\alpha f(x)+\beta f(y)+\epsilon$ (α, β compris entre 0 et 1, $\alpha+\beta=1$). Les auteurs montrent que si E est un ouvert convexe, on peut approcher f à $k_n\epsilon$ près ($k_n=(n^2+3n)/(4n+4)$) par une fonction convexe.

M. Brelot (Grenoble).

Green, John W. Approximately convex functions. Duke Math. J. 19, 499–504 (1952).

Following Hyers and Ulam [see the paper reviewed above], a real-valued function f defined on an n-dimensional convex set S is ϵ-convex if, for every x, y in S and for every λ, $0\leqq\lambda\leqq1$,

$$f(\lambda x+(1-\lambda)y)\leqq\epsilon+\lambda f(x)+(1-\lambda)f(y).$$

The author investigates the continuity properties of ϵ-convex functions and improves a theorem of Hyers and Ulam in the following sense. If f is ϵ-convex and upper (or lower) semi-continuous in S, then there exists a convex function g such that, in S, $g(x)\leqq f(x)\leqq g(x)+p_n\epsilon$. Here p_n depends only on n and is determined by recursion relations. The constant p_n is an improvement on that given by Hyers and Ulam and is shown to be best possible if $n\leqq3$. A slightly weaker theorem is obtained for ϵ-convex functions which are not restricted to be semicontinuous. *F. F. Bonsall.*

Delange, Hubert. Sur une formule de Tchebichef pour le calcul approché des intégrales définies. Acad. Serbe Sci. Publ. Inst. Math. 4, 9–30 (1952).

See the review of the preliminary communication in C. R. Acad. Sci. Paris 231, 602–604 (1950); these Rev. 12, 332. In the meantime V. S. Videnskiĭ has observed [Uspehi Matem. Nauk 7, no. 2(48), 215 (1952)] that the results in question had already been completely obtained by R. O. Kuzmin [C. R. Acad. Sci. Paris 201, 1094–1095 (1935); 202, 272–273 (1936)]. *G. Szegö* (Stanford, Calif.).

Ivanov, V. K. On uniform approximations of continuous functions. Mat. Sbornik N.S. 30(72), 543–558 (1952). (Russian)

Let $f(x)$ and $l_i(x)$ ($i=1, \cdots, n$) be complex-valued continuous functions on the compact set X. Let $P(x)=\sum_1^n \alpha_i l_i(x)$ be the "quasi-polynomial" of best approximation to $f(x)$ and denote by G the set of points where $|P(x)-f(x)|$ attains its maximum E. On G define the function

$$\epsilon(x)=\frac{1}{E}(\overline{P(x)}-\overline{f(x)}) \qquad (|\epsilon(x)|=1).$$

It is shown that this set G has the following important extremum property: For an arbitrary quasi-polynomial $Q(x)=\sum_1^n \beta_i l_i(x)$ there is a point x in G such that

$$\mathrm{Re}\,[\epsilon(x)(Q(x)-f(x))]\geqq E.$$

It is possible that a subset N of G has the same property; such a subset is called a normal set. The main result of the paper is the following: there exists in X always a minimal normal set, and if the number of the points of this set is denoted by $q+1$, then $q\leqq n$ in the real case and $q\leqq2n$ in the complex case. It is shown by examples that the cases $q=n+1$ and $q=2n$ really occur. The approximation problem is called non-singular if $q\leqq n$ and singular if $q>n$; thus, in the real case, the problem is always non-singular. In the non-singular case the problem of best approximation may be reduced to the determination of the maximum of some function, in full analogy to the case of real functions [see de la Vallée Poussin, Leçons sur l'approximation des fonctions d'une variable réelle, Gauthier-Villars, Paris, 1919, chap. VI; and L. G. Šnirel'man, Izvestiya Akad. Nauk SSSR. Ser. Math. 2, 53–59 (1938)]. The singular case yields new difficulties, but it is still possible to reduce the problem of best approximation to finding the maximum, under some supplementary conditions, of a certain function.

B. Sz-Nagy (Szeged).

Bohman, Harald. On approximation of continuous and of analytic functions. Ark. Mat. 2, 43–56 (1952).

Soit $f(x)$ continue pour $0\leqq x\leqq1$, on pose

$$A_n(f)=\sum_{\nu=0}^{n} f(\xi_{\nu,n})\psi_{\nu,n}(x) \quad (n=1, 2, \cdots)$$

où $0\leqq\xi_{\nu,n}\leqq1$, $\xi_{\nu,n}<\xi_{\mu,n}$ si $\nu<\mu$, et où les $\psi_{\nu,n}(x)$ sont des fonctions continues données. Lorsque $\psi_{\nu,n}\geqq0$, pour que

$A_n(f)\to f$ uniformément pour tout f dans $0\leqq x\leqq 1$, il est nécessaire et suffisant que

$$A_n(1)\to 1,\quad A_n(x)\to x,\quad A_n(x^2)\to x^2.$$

Application au cas où $\xi_{\nu,n}=\nu/n$, $\psi_{\nu,n}(x)=e^{-Nx}(Nx)^\nu/\nu!$; il est alors suffisant que $N(n)/n\to 1$, $(n-N(n))/\sqrt{n}\to+\infty$. Considérant alors une fonction analytique $f(z)$ régulière dans la boucle Ω, contenant l'origine, de la courbe $|ze^{1-z}|=1$, continue à l'intérieur et sur le contour de Ω, telle que $f(1)=0$ et que $|f(z)/(1-z)|$ reste borné sur le contour de Ω, alors $A_n[f(z)]\to f(z)$ uniformément dans toute région fermée intérieure à Ω. *J. Favard* (Grenoble).

Hirschman, I. I., Jr., and Widder, D. V. A note on quasi-analytic functions. Acad. Serbe Sci. Publ. Inst. Math. **4**, 57–60 (1952).

Pour que la classe $C(M_n)$ définie par $|f^{(n)}(t)|<Ak^nM_n$, avec $\lim M_n^{1/n}=\infty$, $-\infty<t<\infty$, $n=0,1,2,\cdots$, où A, k dépendent seuls de $f\in C(M_n)$, soit quasi-analytique, il faut et il suffit d'après un théorème de S. Mandelbrojt [Rice Inst. Pamphlet **29**, no. 1 (1942); ces Rev. **3**, 292] que $\sum_{n=0}^{\infty}M_n^e/M_{n+1}^e=\infty$; $M_n^e=\max_{r\geq 0}r^n/T(r)$, $T(r)=\max_{n\geq 1}r^n/M_n$. Elégante démonstration de la condition nécessaire par l'exemple de l'intégrale de Fourier

$$f(t)=\frac{M_0^e}{\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{st}ds}{(s-1)^3\prod_k(1-s/a_k)},\qquad a_k=\frac{M_k^e}{M_{k-1}^e},$$

qui vérifie $|f^{(n)}(t)|<M_n^e$, $f^{(n)}(0)=0$, $f(t)=0$ pour $t>0$, et donne lieu à la remarque: $\limsup|f^{(n)}(t)|^{1/n}<\infty$ pour toute valeur de t (mais non uniformément). *P. Lelong* (Lille).

Hsu, L. C. A theorem concerning an asymptotic integral. Bull. Calcutta Math. Soc. **43**, 109–112 (1951).

By a generalisation of the method of Laplace and Widder [Widder, The Laplace transform, Princeton, 1941; these Rev. **3**, 232] an asymptotic evaluation of the integral

$$I_n=\int_R\varphi(x)f_1(x)f_2(x)\cdots f_n(x)\,dx$$

as $n\to\infty$ is derived, which includes the formula of Laplace-Darboux, and later extensions as special cases. Under suitable conditions

$$I_n\sim\frac{2\Gamma(1/\lambda)\varphi(\xi)f_1(\xi)f_2(\xi)\cdots f_n(\xi)}{\lambda\left\{\dfrac{k_1}{f_1(\xi)}+\dfrac{k_2}{f_2(\xi)}+\cdots+\dfrac{k_n}{f_n(\xi)}\right\}^{1/\lambda}},$$

where $f_n(x)$ attain their positive absolute maximum at $x=\xi$ and where λ is determined by

$$\lim_{x\to\xi}\frac{|f_n(x)-f_n(\xi)|}{|x-\xi|^\lambda}=k_n$$

where $\liminf k_n>0$ and $\limsup k_n<\infty$.

H. A. Lauwerier (Amsterdam).

Theory of Sets, Theory of Functions of Real Variables

Eyraud, Henri. Le continu de seconde classe. Ann. Univ. Lyon. Sect. A. (3) **14**, 5–28 (1951).

Eyraud, Henri. Le Continu Rhodanien des fonctions monotones asymptotiquement nulles. Ses applications à la mesure des croissances et à la théorie des probabilités. Ann. Univ. Lyon. Sect. A. (3) **14**, 119–148 (1951).

These papers are closely related to others by the same author [same Ann. (3) **12**, 53–69 (1949); Leçons sur la théorie des ensembles . . . , 2d ed., Institut de Mathématiques, Lyon, 1949; these Rev. **12**, 166], and are written under the delusion that he has solved the continuum problem. *F. Bagemihl* (Rochester, N. Y.).

Denjoy, Arnaud. Les nodales des suites régulières. C. R. Acad. Sci. Paris **235**, 501–502 (1952).

Viene corretto un errore contenuto alle pp. 456–457 del libro: L'énumération transfinie, livre II, 2ième partie [Gauthier-Villars, Paris, 1952], recentemente pubblicato dall'autore. L'errore riguarda un argomento di dettaglio e d'importanza secondaria rispetto al resto del libro, e non ha perciò conseguenze nel complesso della teoria svolta nel libro stesso. *T. Viola* (Roma).

Kurepa, G. On a characteristic property of finite sets. Pacific J. Math. **2**, 323–326 (1952).

A subset of a partially ordered set (poset) is called a chain if every two of its elements are comparable, an antichain if every two of its elements are incomparable. A poset is said to be ramified if the set of all predecessors of each element of it is a chain. Theorem: a nonnull set is finite if and only if in each ramified partial ordering impressed upon it every maximal chain intersects every maximal antichain. The paper closes with the following two questions: (1) does there exist a nonnull poset such that no antichain intersects every maximal chain? (2) does there exist a nonnull poset such that no chain intersects every maximal antichain? Reviewer's note: there exists a six element poset affirming (2), a simple denumerable, but no finite, poset affirming (1), a not quite so simple, but ramified, denumerable poset affirming both (1) and (2). *W. Gustin* (Bloomington, Ind.).

Kurepa, Djuro. Sur une propriété caractéristique du continu linéaire et le problème de Suslin. Acad. Serbe Sci. Publ. Inst. Math. **4**, 97–108 (1952).

Proof of results announced earlier [C. R. Acad. Sci. Paris **231**, 1113–1114 (1950); these Rev. **12**, 397]. *S. Sherman* (Sherman Oaks, Calif.).

Kolmogorov, A. N. On the foundations of the theory of real numbers. Časopis Pěst. Mat. **76**, 155–157 (1951). (Czech)

Translated from Uspehi Matem. Nauk (N.S.) **1**, no. 1(11), 217–219 (1946).

Schmidt, Jürgen. Beiträge zur Filtertheorie. I. Math. Nachr. **7**, 359–378 (1952).

A filter f over a set E is a nonempty family of subsets of E which, together with X and Y, contains $X\cap Y$ and every subset containing X; in other words, it is an ideal of the boolean ring of subsets of E (the product XY being $X\cup Y$). This paper gives a systematic account of easy results concerning the lattice of all filters over E: distributivity; quotient filter (particular case of the quotient ideals $(A:B)$); principal filters; relations between a filter and the filter of all complements of finite subsets of E; study of the filters the members of which have a given intersection; topological space of ultrafilters; translation of the operations on filters $f, g, \cdots$ in terms of the sets $\Omega_f, \Omega_g, \cdots$ of ultrafilters containing $f, g, \cdots$. *P. Samuel* (Ithaca, N. Y.).

Rohrbach, Hans, und Volkmann, Bodo. Zur Konvergenz von Mengenfolgen. Math. Ann. **124**, 298–302 (1952).

Ostmann gave definitions of convergence and limits for sequences of sets of non-negative integers [Math. Ann. **120**, 165–196 (1948); these Rev. **9**, 499]. The writers prove that

these definitions are equivalent to those given in Hausdorff's "Mengenlehre" [specialized to these particular sets]. This enables them to substitute simpler proofs for some of Ostmann's results. *P. Scherk* (Los Angeles, Calif.).

Ore, Oystein. On the selection of subsequences. Proc. Amer. Math. Soc. **3**, 706–712 (1952).

Let $A=\{a_i\}$ be a sequence of positive numbers tending to infinity with i. Say that $f(x)$ is a selection function for this sequence if $f(x)$ is continuous and monotone increasing for x sufficiently large, say $x>m$, and if it is possible to extract from A an increasing subsequence $B=\{b_i\}$ with $b_i>m$ such that $f(b_i-\delta_i)<b_{i+1}-\delta_{i+1}\leqq b_{i+1}<b_{i+1}+\epsilon_{i+1}<f(b_i+\epsilon_i)$, $i=1, 2, \cdots$, for suitably chosen $\delta_i\geqq 0$ and $\epsilon_i>0$. Say that $f(x)$ is a convergent selection function if also $\lim \delta_i=\lim \epsilon_i=0$. If $f(x)$ is a convergent selection function for the sequence A, then there exists a positive constant k and a subsequence B of A such that $\lim (b_i-f_i(k))=0$ where f_i is the ith iterate of $f(x)$. If for a sequence of integers A, $f(x)$ is a selection function with $\delta_i=0$ and $\epsilon_i<1$ for every i, then there exists a positive constant k such that $[f_i(k)]=b_i$ is a subsequence B of A. These results are a generalization of an idea of W. H. Mills [Bull. Amer. Math. Soc. **53**, 604 (1947); these Rev. **8**, 567] who proved the existence of a constant k such that $[f_i(k)]$ is a subsequence of the prime numbers, where $f(x)=x^3$. The author applies his results to probability sequences, that is, sequences of integers A such that $\lim_{n\to\infty} v(n)/n=p>0$ where $v(n)$ is the number of terms of A which are $\leqq n$. For such a sequence and for any constant $a>1$ there exists a positive constant k such that for sufficiently large i the sequence $B=\{b_i\}$ with $b_i=f_i(k)$, $f(x)=a^x$, is a subsequence of A. *I. Niven* (Eugene, Ore.).

Gelbaum, B. R., and Kalisch, G. K. Measure in semigroups. Canadian J. Math. **4**, 396–406 (1952).

Let S be a semigroup with two-sided cancellation law and with a bounded invariant measure m. The authors prove the following results: 1) if the image of the product $A\times B$ of measurable sets A, B in S by the shearing transformation $(x, y)\to(x, xy)$ is again always a measurable set in $S\times S$ with respect to the completion of the product measure $m\times m$, then S is a group; 2) if S is a locally compact semigroup and m is a topological measure on S, then S is again a group, and if, furthermore, G is commutative or m has the above property in 1), the topology of S can be weakened in such a way that S becomes a compact group whose Haar measure coincides with the completion of m. A counterexample is given to show that a semigroup with a bounded invariant measure is not always a group, and the problem of extending a (not necessarily bounded) measure of a commutative semigroup to a measure of its quotient group is also discussed. *K. Iwasawa* (Cambridge, Mass.).

Kudō, Hirokichi. A theorem of Kakutani on infinite product measures. Nat. Sci. Rep. Ochanomizu Univ. **3**, 10–22 (1952).

The author gives a new proof of Kakutani's theorem on the equivalence of infinite product measures. The method is of independent interest; it is based on a systematic investigation of the set $L(m, m')$ of all those points in the plane which are of the form $(\int f dm, \int f dm')$, where m and m' are probability measures on the same measurable space and where f ranges over all measurable functions from that space to the unit interval. *P. R. Halmos* (Chicago, Ill.).

Kappos, Demetrios A. Über äquimessbare (verteilungsgleiche) Funktionen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 113–128 (1952).

The author studies various conditions on a normalized measure space that ensure the existence of a sequence of (stochastically) independent functions with prescribed distributions. Typical and principal theorem: in the separable case, the distributions can be arbitrarily prescribed if and only if there are no atoms. *P. R. Halmos.*

Kametani, Shunzi. On an elementary treatise of integration. Nat. Sci. Rep. Ochanomizu Univ. **2**, 6–12 (1951).

The author proposes an approach to integration theory in measure spaces, vaguely analogous to the approach to Riemann integration via primitives. The spirit of the author's suggestion is adequately conveyed by the following condensed version of his three theorems. Suppose that S is a totally finite measure space with measure m. If f is a fixed bounded, real-valued, but not necessarily measurable function on S, and if X is a measurable subset of S, write $a(X)$ and $b(X)$ respectively for the infimum and the supremum of f on X. There exists at least one countably additive set function F in S such that $m(X)a(X)\leqq F(X)\leqq m(X)b(X)$ for every measurable set X (Theorem 1), and F is unique if (Theorem 2) and only if (Theorem 3) f is measurable. The proposed treatment appears to be strongly order-theoretic and, therefore, not adaptable to vector-valued integrals. *P. R. Halmos* (Chicago, Ill.).

Goffman, Casper. A generalization of the Riemann integral. Proc. Amer. Math. Soc. **3**, 543–547 (1952).

L'A. presenta una estensione del concetto di integrale, il P-integrale di una funzione $f(x)$ reale e limitata, definita per esempio nell'intervallo $0\leqq x<1$. L'A. suppone che sia assegnata una relazione P fra insiemi S ed intervalli I semiaperti a destra. L'A. scrive $(S, I)^P$ ovvero $(S, I)^{\bar{P}}$ per dire, rispettivamente, che l'insieme S gode o non gode della proprietà P rispetto all'intervallo I e suppone che la proprietà P soddisfaccia alle seguenti condizioni: a) da $(S, I)^P$ e da $T\subset S$ segue $(T, I)^P$; b) da $(S, I)^P$ segue $(CS, I)^{\bar{P}}$, se CS è il complementare di S. Alla funzione $f(x)$ vengono associate le funzioni

$$\varphi(P, f; I)=\inf [y\,|\,(E(f(x)>y), I)^P],$$
$$\psi(P, f; I)=\sup [y\,|\,(E(f(x)<y), I)^P]$$

e l'integrale superiore secondo Burkill della funzione φ e quello inferiore della funzione ψ vengono detti rispettivamente il P-integrale superiore e il P-integrale inferiore della funzione $f(x)$, la quale è P-integrabile se il suo P-integrale inferiore coincide con quello superiore (e in tal caso si parla di P-integrale della f). Se $(S, I)^P$ significa che $S\cap I$ è l'insieme vuoto, il P-integrale coincide con l'integrale secondo Riemann; se $(S, I)^P$ significa che la densità media esterna di S rispetto ad I è minore di $\frac{1}{2}$, il P-integrale di f coincide con l'integrale secondo Lebesgue. Il P-integrale gode della proprietà distributiva, se da $(S, I)^P$ e $(T, I)^P$ segue sempre $(S\cup T, I)^P$; a questo proposito anzi l'A. dimostra un teorema ancora più generale. *G. Scorza Dragoni* (Padova).

***Picone, Mauro, e Viola, Tullio. Lezioni sulla teoria moderna dell'integrazione.** Edizioni Scientifiche Einaudi, Torino, 1952. 404 pp. 5000 lire.

The first 291 pages and eleven chapters of this volume is a printed edition with only minor additions of Picone, Teoria moderna dell'integrazioni della funzioni [Goliandica,

Pisa; these Rev. 13, 729]. The balance of the eleventh chapter contains a discussion of discontinuities of functions of bounded variation according to Vitali, and an integration by parts theorem for Lebesgue-Stieltjes integrals, where f and g are both of bounded variation either in one or two variables, the formulas including the terms which must be added because of common discontinuities of f and g. The final three chapters are due to Viola, and are headed: additive functions of sets, the canonical decomposition of an additive function of sets and the derivation of additive functions. The first of these covers the usual properties of additive functions of sets; the second is concerned with what is usually called the Lebesgue-Nikodým theorem, among other things introduces the Lebesgue definition of integral, and shows its equivalence with the integral introduced in the first part of the book; the final chapter gives a résumé of the theory of derivation over systems of nets after the manner of de la Vallée Poussin. The reviewer noticed the general absence of references in the book and something of a nationalistic slant, though this does not extend to the proof and use of the Vitali covering theorem.

T. H. Hildebrandt (Ann Arbor, Mich.).

De Giorgi, Ennio. Sulla sommabilità delle funzioni assolutamente integrabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 507–510 (1952).

This note shows that if in a space S_r of dimension r there is given an elementary mass $\alpha(T)$ in the sense of Picone (for terminology see the book reviewed above), then a necessary and sufficient condition that the quasicontinuous function $f(p)$ be integrable with respect to α on a Lebesgue set E is that it be possible to find an elementary mass $\beta(T)$ equivalent to $\alpha(T)$ such that $\int_E |f(P)|\,d\nu_\beta$ exist, i.e., $f(P)$ be absolutely integrable with respect to β. Here α and β are equivalent elementary masses if for every closed bounded set C on S_r, and every function $f(P)$ continuous in C, $\int_C f(P)d\alpha = \int_C f(P)d\beta$.

T. H. Hildebrandt.

Aquaro, Giovanni. Sopra un teorema di H. Lebesgue. Portugaliae Math. **11**, 75–88 (1952).

This note gives a demonstration of the theorem of Lebesgue on the derivation of function of sets, without using the Vitali covering as is done by Lebesgue, nor the integral representation of an absolutely continuous function of sets as is done by de la Vallée Poussin. The notion of derivation with respect to a net of cubes developed by de la Vallée Poussin plays a prominent role in the demonstration.

T. H. Hildebrandt (Ann Arbor, Mich.).

Nevanlinna, Rolf. Beweis des Satzes über die Vertauschbarkeit der Differentiationen. Math. Z. **56**, 120–121 (1952).

Ważewski, T. Sur une condition nécessaire et suffisante pour qu'une fonction continue soit monotone. Ann. Soc. Polon. Math. **24** (1951), 111–119 (1952).

L'A. dimostra che la funzione $f(x)$, continua nell'intervallo Δ, è decrescente ivi in senso lato se trasforma in un insieme di misura nulla l'insieme dei punti di Δ nei quali il suo numero derivato destro inferiore è positivo; e di qui trae un corollario immediato, relativo al caso che la funzione $f(x)$ soddisfaccia alla condizione N di Lusin, trasformi cioè in insiemi di misura nulla gli insiemi di misura nulla dell'intervallo Δ.

G. Scorza Dragoni (Padova).

Homma, Tatsuo. A theorem on continuous functions. Kōdai Math. Sem. Rep. **1952**, 13–16 (1952).

The author calls "α-function" a function $f(x)$ continuous in $[0, 1]$, such that $f(0)=0$, $f(1)=1$, $0 \leqq f(x) \leqq 1$ in $[0, 1]$, and he proves the theorem: If $f(x)$ and $g(x)$ are nowhere constant α-functions, then there exist α-functions $\varphi(x)$ and $\psi(x)$ such that $f(\varphi(x)) = g(\psi(x))$ for every $x \in [0, 1]$. The condition that $f(x)$ and $g(x)$ are nowhere constant is essential, which is shown by a simple example of T. Minagawa. The author generalizes his theorem also to the case of more than two functions.

A. Rosenthal (Lafayette, Ind.).

Gahariya, K. K. The representation of a function of two variables at the Lebesgue points by singular double integrals. Soobščeniya Akad. Nauk Gruzin. SSR. **12**, 257–264 (1951). (Russian)

The author first deduces from a known theorem on differentiability of double integrals [cf. S. Saks, Theory of the integral, 2nd ed., Stechert, New York, 1937, p. 148] the following result (Theorem 1): if $f(x, y) \log^+ |f(x, y)|$ is summable over the rectangle $R[a, b; c, d]$ then almost all points of R are Lebesgue points of $f(x, y)$, that is, points (x, y) such that

$$\lim_{h,k\to 0} \frac{1}{hk} \int_x^{x+h} \int_y^{y+k} |f(r, t) - f(x, y)|\,dr dt = 0.$$

Secondly, let $\Phi_{mn}(r, t, x, y)$ $(m, n = 1, 2, \cdots)$, defined on $R \times R$, be a kernel, that is, let the double integral of $\Phi_{mn}(r, t, x, y)$, over any subrectangle $[\alpha, \beta; \gamma, \delta]$ of R in which (x, y) is a fixed interior point, tend to 1 as $m, n \to \infty$. Also, let $\Psi_{mn}(r, t, x, y)$ be a "hump-backed" majorant of $\Phi_{mn}(r, t, x, y)$, that is, let $|\Phi_{mn}(r, t, x, y)| \leqq \Psi_{mn}(r, t, x, y)$, and let $\Psi_{mn}(r, t, x, y)$ decrease when $|r-x|$ increases and when $|t-y|$ increases. Then (Theorem 2) if

$$\int\int_R \Psi_{mn}(r, t, x, y)\,dr dt < K(x, y),$$

$$\int_a^b \Psi_{mn}(r, y, x, y)\,dr < K_1(x, y),$$

and $\int_c^d \Psi_{mn}(x, t, x, y)\,dx < K_2(x, y)$, where K, K_1, K_2 depend only on x, y, and if $a < x < b$, $c < y < d$, we must have

$$\lim_{m,n\to\infty} \int\int_R \Phi_{mn}(r, t, x, y) f(r, t)\,dr dt = f(x, y),$$

for any summable function $f(r, t)$ for which (x, y) is a Lebesgue point. The author shows that this theorem is applicable, in particular, to the kernel

$$[\sin^2 \tfrac{1}{2}m(r-x) \sin^2 \tfrac{1}{2}n(t-y)] / [4\pi^2 mn \sin^2 \tfrac{1}{2}(r-x) \sin^2 \tfrac{1}{2}(t-y)],$$

which arises in connexion with Cesàro summation of double Fourier series: using also his Theorem 1 he deduces (Theorem 3) that the double Fourier series of $f(x, y)$ is summable $C_{1,1}$ almost everywhere in R if $f(x, y) \log^+ |f(x, y)|$ is summable over R, a result first obtained by Zygmund [Fund. Math. **25**, 234 (1935)].

H. P. Mulholland (Birmingham).

Caccioppoli, Renato. Misura e integrazione sulle varietà parametriche. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 365–373 (1952).

In this second note [for part I see same Rend. (8) **12**, 219–227 (1952); these Rev. **13**, 925] the author sketches, some-

what cavalierly and only in the case of parametric surfaces, further definitions and properties, which he believes to be obviously generalizable to varieties of higher dimension and which include multiplicity and total variation of the projection of a parametric surface on a plane and their semi-continuity. The author notes the agreement of his results with those obtained by others, who make use of deep topological methods, and he stresses again the simplicity and naturalness of his own treatment, in which the reviewer regretfully finds again no more than a misleading plausibility.

L. C. Young (Madison, Wis.).

Caccioppoli, Renato. Misura e integrazione sulle varietà parametriche. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 629–634 (1952).

This third and final note treats mainly the formulae of Gauss and Stokes in what the author terms their natural domain of validity. It is thus clear that the reader must expect no assistance in the heavy task of clarifying the previous plausible proofs and definitions. Except for an unexpected admission that there are technical difficulties, perhaps minor ones, in the way of certain further desirable developments, this final note is written in the same style as its predecessors [see the preceding review and reference cited there].

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

***Heinhold, Josef. Theorie und Anwendung der Funktionen einer komplexen Veränderlichen. Ein Lehrbuch für Studierende der Naturwissenschaften und Technik. Erster Band.** Leibniz Verlag, München, 1949. 213 pp. 15 DM.

Turán, Pál. Some recent applications of a method in analysis. Magyar Tud. Akad. Mat. Fiz. Ozst. Közleményei **2** (1951), 145–153 (1952). (Hungarian)

The author discusses some new applications of his method [see, e.g., Časopis Pěst. Mat. Fys. **74** (1949), 123–131 (1950); these Rev. **12**, 490] which he used successfully in recent years in analysis and number theory. He outlines applications in the theory of differential equations, and function theory. All these will be given in more detail in a forthcoming book of the author.

P. Erdös.

San Juan, Ricardo. On the existence of a holomorphic function which approximates asymptotically with assigned bounds to a given series, and of a real function infinitely differentiable in an interval with assigned derivatives in a point and bounds in the interval. Collectanea Math. **4**, 83–91 (1951). (Spanish)

The author indicates necessary and sufficient conditions in order that there exist a function $f(z)$ holomorphic in a region R, with $z=0$ as frontier point, and satisfying there the relationships $|f(z)-\sum_{\nu=0}^{n-1}a_\nu z^\nu| \leqq m_n|z|^n$ $(n \geqq 1)$ (the conditions relating the elements R, $\{a_n\}$, $\{m_n\}$). He gives also necessary and sufficient conditions in order that there exist an infinitely differentiable function $f(x)$ on $[a, b]$ with $f^{(n)}(x_0)=a_n$ $(x_0 \in [a, b], n \geqq 0)$, $|f^{(n)}(x)| \leqq m_n$ $(n \geqq 1)$.

S. Mandelbrojt (Houston, Tex.).

Smith, R. C. T. An interpolatory function analogous to the cardinal function. Quart. J. Math., Oxford Ser. (2) **3**, 235–240 (1952).

The author considers the series

$$G(z)=-\frac{1}{\Gamma(-z)}\sum_{r=0}^{\infty}\frac{(-1)^r b_r}{r!(z-r)},$$

which (if convergent) represents an entire function taking values b_r at the nonnegative integers r. As an analogue to the "consistency" property of the cardinal series for interpolation at all the integers [cf., J. M. Whittaker, Interpolatory function theory, Cambridge Univ. Press, 1935, p. 68], he proves that if either $\sum|b_r/r!|^p<\infty$ $(p>1)$ or $b_r/r!=o(r^\delta)$ $(\delta>0)$ and if $\Re(\lambda)<0$, then

$$G(x)=-\frac{1}{\Gamma(\lambda-x)}\sum_{n=0}^{\infty}\frac{(-1)^n G(n+\lambda)}{n!(x-n-\lambda)}.$$

Further properties of the series are discussed.

R. P. Boas, Jr. (Evanston, Ill.).

***Berghuis, J. A class of entire functions used in analytic oscillatory interpolation.** Computation Department, Mathematical Centre, Amsterdam, Rep. **R 143,I** (1952). i+13 pp.

***Berghuis, J. A note on entire functions used in analytic interpolation.** Computation Department, Mathematical Centre, Amsterdam, Rep. **R 143,II** (1952). i+4 pp.

Let $u_n=(u+n)\pi$, $v_n=(\sin u_n)/u_n$. The papers are concerned with properties of the integral functions

$$F_k(t, u)=\sum v_n^k \exp(-tu_n^2/\pi),$$

where the summation is on n from $-\infty$ to $+\infty$. The special case $t=0$ has previously been discussed by I. J. Schoenberg [Quart. Appl. Math. **4**, 112–141 (1946); these Rev. **8**, 55] in connection with his studies of osculatory interpolation. The second paper also considers the function $G_k(t, u)$ in which an additional $(-1)^n$ appears in the above summand. $F_k(t, u)$ has a period 1 and is even in u. Integral representations of F_k and G_k are obtained and the product expansion of the first of these functions is found. The distribution of the zeros of these two functions is studied, particularly for $k=0$, 1, and 2. Finally, some inequalities are given which follow from the fact that $F_k(t, iy)>0$ for y real, and similarly for G_k. [The proofreading of these mimeographed reports has been careless, but most of the errors are obvious. However, the error in the title of the first of these two papers, which should read "osculatory" rather than "oscillatory", might be misleading.]

P. W. Ketchum (Urbana, Ill.).

Mursi, Zaki. On an interpolated integral function of given order. Euclides, Madrid **12**, 317–328 (1952).

The author constructs, of course under suitable restrictions on the data, an entire function $f(z)=g(z)h(z)$ of finite order which satisfies $f^{(n)}(a_n)=b_n$ with given points a_n and values b_n; $g(z)$ is a canonical product with an $(n+1)$-fold zero at a_n and $h(z)$ is a meromorphic function with an $(n+1)$-fold pole at a_n. Although constructions for $g(z)$ and $h(z)$ are well known, this seems to be the first time that a solution $f(z)$ of finite order has been given for the author's problem under fairly general hypotheses.

R. P. Boas, Jr. (Evanston, Ill.).

Macintyre, Sheila Scott. Some generalizations of two-point expansions. Proc. Cambridge Philos. Soc. **48**, 583–586 (1952).

Given real numbers a and b, $a \neq b$, there exist polynomials $P_n(z, a, b)$ such that one has the formal expansion

$$(1)\quad F(z) = \sum_0^\infty P_n(z, a, b) F^{(2n)}(a+2n) + \sum_0^\infty P_n(z, b, a) F^{(2n)}(b+2n).$$

The polynomials $P_n(z)$ may be obtained from the conditions: $P_0(z) = (z-b)/(a-b)$, $P_n(a) = P_n(b) = 0$, $P_n''(z) = P_{n-1}(z-2)$, $n \geqq 1$. Following standard procedures [see Buck, Trans. Amer. Math. Soc. **64**, 283–298 (1948); these Rev. **10**, 693], the author applies (1) to the function $F(z) = e^{tz}$, and extends it to a class of entire functions of exponential type by means of the Pólya integral representation. If a and b are close together, $|a-b| < 9.957\cdots$, then (1) is valid for the same class of functions $F(z)$ as that for the Abel series [loc. cit., p. 293], in particular for functions of type $A < .278$. If $|b-a| = \pi/\theta \geqq 9.957$, then a smaller class of functions is obtained. The critical type A is a root of the equation $\theta \sec\theta \exp\theta \tan\theta$. If $F(z)$ is not entire, then (1) cannot hold, since the sum of the series, when it converges, is entire. In this case, denote the sum by $A(z)$. Following methods similar to those of a previous paper [Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 222–231 (1952); these Rev. **13**, 738], the author shows that as $x = R(z)$ increases,

$$F(z) - A(z) = O(1/x \log x)^k$$

where $F(z)$ is regular for $|\theta| < \beta$, $\pi/2 < \beta < 3\pi/4$ and for $x > -h$, and in this sector obeys the Abel series growth restriction. Similar results may be obtained for the series

$$F(z) = \sum A_n(z) F^{(2n)}(\lambda n) + \sum B_n(z) F^{(2n+1)}(\lambda n + b)$$

which reduces to Abel's series for $b = 0$, $\lambda = 1$, to a series of Lidstone for $b = 1$, $\lambda = 0$, and to Taylor's series if $\lambda = b = 0$.

R. C. Buck (Madison, Wis.).

Macintyre, A. J. Asymptotic paths of integral functions with gap power series. Proc. London Math. Soc. (3) **2**, 286–296 (1952).

Pólya has investigated the possibility of asymptotic paths of entire functions (*) $F(z) = \sum c_n z^{\lambda_n}$ with gap power series [Math. Z. **29**, 549–640 (1929)]. This paper gives further results along the same lines. Pólya's result that if $\lambda_n \sim n \log\log n$, then there exists an entire function (*) for which $F(x) \to 0$ as $x \to \infty$ is extended to include all sequences $\{\lambda_n\}$ for which $\sum 1/\lambda_n = \infty$. For λ_n/n bounded the $F(z)$ constructed are of finite order. Now let $F(z)$ be an entire function (*) of finite order and let $\lambda_n/n \to \infty$. Pólya conjectured that in this case the upper limit of $\log m(r)/\log M(r)$ is equal to 1. The present author shows that at least the following is true. For every $\epsilon > 0$ there are arbitrarily large r such that the annulus $r(1-\epsilon) < |z| < r$ contains a curve enclosing the origin on which $\log|F(z)|/\log M(r) > 1-\epsilon$.

J. Korevaar (Madison, Wis.).

Tricomi, Francesco G. A new entire function related to a well-known noncontinuable power series. Comm. Pure Appl. Math. **5**, 213–222 (1952).

The author investigates the entire function

$$G(z) = \sum_{n=1}^\infty (a^n - 1)^{-1} z^n/n! = \sum_{m=1}^\infty \{\exp(za^{-m}) - 1\} \quad (a > 1).$$

He was led to this function by a study of the partial sums $F_n(z)$ of the series $F(z) = \sum_{n=0}^\infty \exp(a^n z)$ ($\mathrm{Re}\, z < 0$) which satisfy the relation $F_n(z) - m = G(a^n z) - G(z)$. The present investigation is centered upon the auxiliary functions $K(u) = -zG'(z)$ and $\Phi(u) = -zF'(z)$ where $z = -a^u$ and, in the case of $\Phi(u)$, $|\mathrm{Im}\, u - 2k\pi A| < \frac{1}{2}\pi A$, $A = 1/\log a$. Various methods of attack are indicated: the difference equation $\Delta K(u) = f(u)$; the Fourier transform $\int_{-\infty}^{\infty} K(u) e^{iuv} du$ which is calculated for $0 < \mathrm{Im}\, v < 1/A$; and so on. Asymptotic estimates for $K(u)$ as $\mathrm{Re}\, u \to \infty$ are obtained as follows. One may write $K(u) = \sum_{m=1}^\infty f(u-m)$, $\Phi(u) = \sum_{m=0}^\infty f(u+m)$, where $f(u) = a^u \exp(-a^u)$. The Fourier series of the doubly periodic function $P(u) = K(u) + \Phi(u)$ belonging to its real period 1 leads to estimates for $P(u)$ on and near the real axis ($P(u)$ is practically constant there provided $a > 1$ is not too large) from which the results for $K(u)$ follow.

J. Korevaar (Madison, Wis.).

Sheffer, I. M. On certain entire functions. Proc. Amer. Math. Soc. **3**, 514–516 (1952).

Let $f(z)$ be an integral function such that $f^{(n)}(0)$ ($n = 0, 1, 2, \cdots$) takes at most N distinct values $a_1, a_2, \cdots$, and let $\sigma = \min\{(a_p - a_q)/(a_r - a_s)\}$. It is shown that there exists an integer $N = N(k, \sigma, A)$ such that if the sequence $f^{(n)}(z_0)$ has more than N equal values and $0 < |z_0| < A$, then $f(z)$ is the sum of a polynomial and an exponential sum $\sum A_j \exp(\omega^j z)$ ($\omega^m = 1$).

A. J. Macintyre.

Arima, Kihachiro. On the zeros of integral functions of integral order. J. Math. Soc. Japan **4**, 67–69 (1952).

Let $f(z)$ be an entire function of integral order ρ. The author proves that (1) if $\log\log M(r)/\log r \to \rho$, then $\log n(r, \alpha)/\log r$ has a limit (independent of α) except perhaps for a set of α of inner logarithmic capacity zero; (2) if $\log M(r)/r^\rho$ is bounded from 0 and ∞, so is $n(r, \alpha)/r^\rho$ with a possible exceptional set of the same kind. With possibly larger exceptional sets these results were proved by R. C. Young [Valiron, Lectures on the general theory of integral functions, Toulouse, 1923, p. 189]. It is known that the exceptional sets in both cases may be uncountable.

R. P. Boas, Jr. (Evanston, Ill.).

Melman, N. N. Differential inequalities and some questions of the distribution of zeros of entire and single-valued analytic functions. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 3–62 (1952). (Russian)

This is a unified exposition of results, mostly recent, dealing with inequalities of the form "$|f(x)| \leqq |\omega(x)|$ implies $|f'(x)| \leqq |\omega'(x)|$," and their generalizations, with entire functions whose zeros are confined to a half-plane, and with limits of sequences of polynomials whose zeros are on a line or in a half-plane. Among other things, the author gives an account of work of Pólya and Schur [J. Reine Angew. Math. **144**, 89–113 (1914)]; he proves theorems of B. Levin [Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 45–84 (1950); these Rev. **11**, 510] and theorems of his own which have been given, with less detailed proofs, in a series of notes and reviewed in detail [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 46–49, 179–181 (1943); Doklady Akad. Nauk SSSR (N.S.) **71**, 609–612 (1950); **82**, 185–188 (1952); these Rev. **6**, 59; **11**, 509; **13**, 732].

R. P. Boas, Jr.

Tsuji, Masatsugu. Wiman's theorem on integral functions of order $<1/2$. Proc. Japan Acad. **26**, nos. 2–5, 117–130 (1950).

The author defines upper strong logarithmic density for a linear set E as

$$\limsup_{r/a \to \infty} \frac{1}{\log(r/a)} \int t^{-1} dt,$$

where the integral extends over the part of E in (a, r); and similarly for lower strong logarithmic density. The upper strong logarithmic density does not exceed the upper density and is not less than the upper logarithmic density (upper logarithmic length). Let $f(z)$ be an entire function, with maximum modulus $M(r)$ and minimum modulus $m(r)$. It was known that if $f(z)$ is of order ρ, $0<\rho<\frac{1}{2}$, then the set where $\log m(r)>r^{\rho-\epsilon}$, $\epsilon>0$, has upper density at least $1-2\rho$ [Besicovitch, Math. Ann. 97, 677–695 (1927); Pennycuick, J. London Math. Soc. 10, 210–212 (1935)]; the author proves the stronger result that upper density can be replaced by upper strong logarithmic density; he also shows that it cannot in general be replaced by upper logarithmic density, and that if $f(z)$ is of infinite type of order ρ, $r^{\rho-\epsilon}$ can be replaced by kr^{ρ} for any positive k.

Now let D be a domain containing $z=0$ and with ∞ on its boundary, D_r the part of D in $|z|<r$, D_r^0 the connected subdomain of D_r containing $z=0$, θ_r the part of its boundary on $|z|=r$, $r\theta(r)$ the length of the maximum arc of $|z|=r$ in θ_r, and $\bar{\theta}(r)=\theta(r)$ or ∞ according as $|z|=r$ meets the boundary of D or not. The author proves several more theorems of which the following are representative. Let $f(z)$ be an entire function of finite order ρ. Let D be a domain on whose boundary $|f(z)|\leqq r^k$; then

$$\limsup \frac{2\pi}{\log r}\int_1^r \frac{dr}{r\bar{\theta}(r)}\leqq 2\rho,$$

and as a consequence the lower logarithmic density of the set where $m(r)>r^k$ is at least $1-2\rho$ if $\rho<\frac{1}{2}$. However, if $\phi(r)$ increases and $\limsup \phi(r)/\log r=\infty$, there is an $f(z)$ of order ρ, for any ρ in $0<\rho<1$, such that the lower logarithmic density of the set where $m(r)>e^{\phi(r)}$ is zero. Let D be a domain on whose boundary $\log|f(z)|\leqq r^{\rho-\epsilon}$; then

$$\liminf_{r/a\to\infty} \frac{2\pi}{\log(r/a)}\int_a^r \frac{dr}{r\bar{\theta}(r)}\leqq 2\rho.$$

Hence if Λ_θ is the part of the ray $\arg z=\theta$ where

$$\log|f(z)|\leqq r^{\rho-\epsilon},$$

then Λ_θ has lower strong logarithmic density at most 2ρ. The set where $\log M(r)>r^{\rho-\epsilon}$ has upper strong logarithmic density 1, but this is not necessarily true for upper logarithmic density. *R. P. Boas, Jr.* (Evanston, Ill.).

Schubart, Hans, und Wittich, Hans. Einige ganze Funktionen und ihre Riemannschen Flächen. Math. Ann. 124, 450–452 (1952).

$g(z)$ denotes the entire function $\prod_{n=1}^{\infty}(1-z/n^l)$, $l>1$, of order $1/l$ and mean type. The zeros of $g'(z)$ are denoted by ξ_k. For $1<l\leqq 2$, $|g(\xi_k)|<1/\pi k$. For $l>2$ the following conjecture of the first author [Math. Ann. 124, 55–64 (1951); these Rev. 13, 643] is proved:

$$\lim_{k\to\infty}\frac{\log\log|g(\xi_k)|}{\log \xi_k}=1/l,\quad \limsup_{k\to\infty}\frac{\log|g(\xi_k)|}{\xi_k^{1/l}}=A,\quad 0<A<\infty.$$

Thus for $l>2$ the function $g(z)$ may be used as an illustration of Wiman's theorem for entire functions of order $<\frac{1}{2}$. *J. Korevaar* (Madison, Wis.).

Potugina, I. V. On estimation of coefficients of odd univalent functions. Doklady Akad. Nauk SSSR (N.S.) 85, 1215–1217 (1952). (Russian)

If $f(z)=z+\sum_{k=1}^{\infty}c_{2k+1}z^{2k+1}$ is univalent in $|z|<1$, then $|c_{2k+1}|<2.78$, ~~$n=3, 5, \cdots$~~ $k=1, 2, \cdots$. *A. W. Goodman.*

Erwe, Friedhelm. Über die Schlichtheitsschranken gewisser Funktionenfamilien. Math. Z. 56, 57–64 (1952).

Let $F_{k,a}$ ($k>0$, $a>0$) represent the family of meromorphic functions $\varphi(z)=z+a_1/z+a_2/z^2+\cdots$ in $|z|>1$ such that $\sum_{\nu=1}^{\infty}\nu|a_\nu|^k\leqq a$. Let $\sigma(\varphi)$ be the inf of those numbers r such that φ is schlicht in $|z|>r$, and let $S(F_{k,a})$ be the sup of $\sigma(\varphi)$ for $\varphi\in F_{k,a}$. If $k>1$, then

$$S(F_{k,a})=\left(1+\frac{1}{a^{2(k-1)}}\right)^{(k-1)/k},$$

and this bound is attained for the function

$$\varphi(z)=z+\frac{a^{1/(k-1)}\epsilon}{M(Mtz-1)},\quad M=1+a^{1/2(k-1)}.$$

If $k\leqq 1$, $a>1$, then $S(F_{k,a})=a^{1/2k}$ and this bound is attained for $\varphi(z)=z+a^{1/k}\epsilon^2/z$. If $k\leqq 1$, $a\leqq 1$, then $S(F_{k,a})=1$, and this bound is attained for all φ in $F_{k,a}$. Similar results are obtained for the class $H_{k,a}$ of functions $\varphi(z)=z+\beta_2z^2+\beta_3z^3+\cdots$ in $|z|<1$ for which $\sum_{\nu=2}^{\infty}\nu|\beta_\nu|^k\leqq a$. *G. Springer.*

Umezawa, Toshio. A class of multivalent functions with assigned zeros. Proc. Amer. Math. Soc. 3, 813–820 (1952).

En relation avec des notions introduites par Robertson [Amer. J. Math. 58, 465–472 (1936); Ann. of Math. (2) 38, 770–783 (1937); Duke Math. J. 5, 512–519 (1939); ces Rev. 1, 9], Goodman [Trans. Amer. Math. Soc. 68, 204–223 (1950); Proc. Amer. Math. Soc. 2, 349–357 (1951); ces Rev. 11, 508; 13, 22], Goodman et Robertson [Trans. Amer. Math. Soc. 70, 127–136 (1951); ces Rev. 12, 691], l'auteur étudie certaines classes de fonctions multivalentes de zéros donnés. Si $w=f(z)$ est holomorphe pour $|z|\leqq 1$ et a $p\geqq 0$ zéros dans ce cercle, il existe un ζ, $|\zeta|=1$, tel que $\arg f(-\zeta)=\arg f(\zeta)+p\pi$ [cf. de Bruijn, Nederl. Akad. Wetensch., Proc. 44, 47–49 (1941); ces Rev. 2, 274], la droite $(0, f(\zeta))$ du plan w est alors appelée un diamètre de $f(z)$. La définition d'une direction d'étoilement d'ordre p, issue de l'origine étant celle de Robertson, l'auteur considère la classe $D(p)$ des fonctions w pour lesquelles une direction d'étoilement d'ordre p est aussi un diamètre. Il établit notamment que: Si $f(z)=z^q+\sum_{q+1}^{\infty}a_nz^n$ est de la classe $D(p)$ et a $p-q$ autres zéros β_j, $0<|\beta_j|<1$, on a

$$|a_n|\leqq B_n,\quad n\geqq q+1,\quad |f(re^{i\theta})|\leqq F(r),$$

où $F(z)$ est la fonction introduite par Goodman

$$F(z)=\frac{z^q}{(1-z)^{2p}}\prod_{j=1}^{p-q}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|),$$

et les B_n ses coefficients. Il étudie d'une façon analogue les fonctions $f(z)$ qui sont telles que $zf'(z)$ appartienne à la classe $D(p)$, puis des sous-classes de ces fonctions. *G. Valiron* (Paris).

Collingwood, E. F., and Cartwright, M. L. Boundary theorems for a function meromorphic in the unit circle. Acta Math. 87, 83–146 (1952).

Les auteurs étudient d'une façon approfondie le comportement d'une fonction $w=f(z)$ méromorphe pour $|z|<1$, dans le voisinage de la circonférence $|z|=1$. Dans la première partie, il s'agit du comportement global. Adoptant la terminologie de Seidel [Trans. Amer. Math. Soc. 34, 1–21 (1932); 36, 201–226 (1934)] ils introduisent le cluster set (domaine d'indétermination de Painlevé) $C(f)$: $a\in C(f)$ s'il existe une suite z_n, $|z_n|<1$, $\lim|z_n|=1$, telle que $\lim f(z_n)=a$; l'ensemble limite $R(f)$: $a\in R(f)$ s'il existe une

suite z_n, $|z_n|<1$, $\lim|z_n|=1$, telle que $f(z_n)=a$; l'ensemble des valeurs asymptotiques $\Gamma(f)$: $a \in \Gamma(f)$ s'il existe un chemin asymptotique continu $z=z(t)$, $0<t<1$, $|z(t)|<1$, $\lim_{t=1}|z(t)|=1$, $f(z(t))=a$. $\mathfrak{C}E$ désigne le complémentaire, $\mathfrak{F}E$ la frontière, $\bar{E}$ la fermeture d'un ensemble E. Ils introduisent aussi la fonction $T(r, f)$ de Nevanlinna, montrent que $C(f)$ est un continuum, rappellent les théorèmes de Schottky, Nevanlinna, et Frostman. L'ensemble limite sur $|z|=1$ d'un chemin asymptotique peut être un point P ou un arc non nul A, d'où les ensembles $\Gamma_P(f)$, $\Gamma_A(f)$, et $\Gamma_S(f)$ si $|\arg z(t)|$ n'est pas borné ($\Gamma_S(f)$ peut exister d'après Valiron [C. R. Acad. Paris **198**, 2065–2067 (1934)]). Des théorèmes du type de Fatou-Riesz-Nevanlinna, les auteurs déduisent que, si $T(r, f)$ est borné, $\Gamma_A(f)$ est vide. Un théorème de Koebe les conduit à considérer l'ensemble $\phi(f)$ des a tels que $f(z)-a$ tende vers 0 sur une suite d'arcs tendant vers un arc non nul de $|z|=1$; ils montrent notamment que $\phi(f)$ est vide si $\mathfrak{C}R(f)$ contient plus de deux valeurs. Les relations de $\Gamma(f)$ avec les propriétés de la fonction inverse $z=z(w)$ sont alors étudiées; les ensembles $\Omega(f)$ des singularités transcendantes des branches intérieures de $z(w)$ et $\Pi(f)$ des valeurs de $f(z)$ aux points réguliers de $|z|=1$ sont définis; on a $\Gamma(f)=\Omega(f)\cup\Pi(f)$. Lorsque $\mathfrak{C}C(f)$ n'est pas vide, on se ramène au cas des fonctions bornées qui est alors considéré, ce qui montre que, dans ce cas, $\Gamma(f)=\Gamma_P(f)$ est de mesure linéaire positive.

Une suite de lemmes conduit au théorème principal de cette étude globale: (i) si $\Gamma(f)$ est quelconque,

$$\mathfrak{F}R(f)\cup\mathfrak{F}C(f)=\overline{\mathfrak{C}R}(f)\cap C(f)\subseteq\bar{\Gamma}(f);$$

(ii) si $\Gamma(f)$ est de mesure linéaire nulle, $\mathfrak{C}R(f)\subseteq\Gamma(f)$. Un exemple déduit de la théorie des fonctions automorphes montre que la proposition (i) est dans un certain sens la meilleure possible. Une série de corollaires du théorème principal et des lemmes préliminaires est donnée; entre autres: si $T(r, f)$ n'est pas bornée, $\mathfrak{C}R(f)\subseteq\bar{\Gamma}(f)$; si $\Gamma(f)$ est vide, $\mathfrak{C}R(f)$ est vide (Noshiro); si $\mathfrak{C}R(f)$ est de capacité positive, $\Gamma(f)$ est de capacité positive; si $\mathfrak{C}R(f)$ est de mesure linéaire positive, $\Gamma(f)$ est de mesure linéaire positive.

Dans une seconde partie, les auteurs étudient les ensembles analogues à $C(f)$, $R(f)$, $\Gamma(f)$ lorsqu'on se borne à prendre z au voisinage d'un point $e^{i\theta}$ de $|z|=1$; ces ensembles sont appelés $C(f, e^{i\theta})$, etc. Les théorèmes du type de Picard sont remplacés par ceux de Gross-Iversen. Le théorème principal analogue à celui de l'étude globale est établi par une marche analogue, il comporte des corollaires correspondants. On rejoint ici en quelques points des études de l'école japonaise de Noshiro, en particulier on retrouve un théorème de Ohtsuka [J. Math. Soc. Japan **2**, 1–15 (1950), pp. 1–3; ces Rev. **13**, 336]; les auteurs, suivant une méthode de Doob [Ann. of Math. **33**, 753–757 (1932)], généralisent le théorème de Gross-Iversen.

La troisième partie donne une classification des singularités sur $|z|=1$ et une étude de leur distribution. $H(f)$ est l'ensemble des points $e^{i\theta}$ pour lesquels $\mathfrak{C}R(f, e^{i\theta})$ est de capacité nulle, $F_1(f)$ l'ensemble des $e^{i\theta}$ en lesquels l'ensemble $F(f)$ des points de Fatou est de densité 1. Une nouvelle démonstration du théorème de Littlewood: tout point de $|z|=1$ appartient à $H(f)$ ou à $F_1(f)$, est donnée; un théorème de Plessner [J. Reine Angew. Math. **158**, 219–227 (1927)] est établi par les méthodes des auteurs; les points appelés points de Picard par Valiron [C. R. Acad. Sci. Paris **186**, 935–936 (1928)] sont aussi examinés.

Dans un appendice, les résultats obtenus sont étendus au cas des fonctions méromorphes dans un domaine de Jordan.

G. Valiron (Paris).

Carleson, Lennart. Sets of uniqueness for functions regular in the unit circle. Acta Math. **87**, 325–345 (1952).

Suppose $f(z)=a_0+a_1z+\cdots$ is regular in $|z|<1$ and continuous in $|z|\leqq 1$. A set E in $(0, 2\pi)$ is defined to be a set of uniqueness for a class of such functions if $f(e^{i\theta})=0$ for θ in E implies $f(z)\equiv 0$. We may assume E closed. Let E_t be the set of points distant at least t from E, l_n the lengths of the complementary intervals of E. Then the equivalent conditions

$$(1)\qquad \int_0^1 m(E_t)t^{-1}dt=\infty$$

or

$$(2)\qquad m(E)>0 \quad\text{or}\quad \sum_{n=1}^{\infty} l_n\log l_n^{-1}=\infty$$

are shown to be necessary and sufficient for E to be a set of uniqueness when either $f(z)$ satisfies a Lipshitz condition of some order α or $|a_n|=O(n^{-p})$ for a fixed $\alpha>0$ or $p>1$. The direct part of this result is due to Beurling [Acta Math. **72**, 1–13 (1939); these Rev. **1**, 226]. If $|a_n|=O[\exp(-n^p)]$ a sufficient condition is

$$\int_0^1 m(E_t)t^{-1/(1-p)}dt=\infty, \quad 0<p<\tfrac{1}{2}.$$

If $|a_n|\leqq\rho_n$, where $\log\rho_n^{-1}$ is concave and

$$\sum n^{-3/2}\log\rho_n=-\infty,$$

then it is sufficient for E to be infinite. Note that the condition (2) is satisfied by some countable sets but not satisfied by all Cantor sets of dimension 1. If $f(z)$ is to possess a bounded Dirichlet integral, sets of uniqueness have positive capacity. Subject to a certain homogeneity condition on E (which in particular excludes sets of zero capacity), (1) is again necessary and sufficient for uniqueness. For the class of absolutely convergent Fourier series sets of uniqueness having zero Lebesgue measure are constructed. A set of uniqueness may be a countable union of sets that are not.

W. K. Hayman (Exeter).

Royster, W. C. Convexity and starlikeness of analytic functions. Duke Math. J. **19**, 447–457 (1952).

Soit $w=f(z)$ une fonction analytique de la variable complexe z dans un domaine contenant une courbe Γ, $z=x(t)+iy(t)$ où $x(t)$ et $y(t)$ sont deux fois dérivables et $z'(t)\neq 0$, et soit C la transformée de Γ par $w=f(z)$. La fonction $f(z)$ est dite convexe sur Γ si, ψ étant l'angle avec l'axe réel de la tangente à C au point t, on a $d\psi/dt\geqq 0$. Si $w\neq w_0$, $\phi=\arg(w-w_0)$ et si $d\phi/dt\geqq 0$, $f(z)$ est dite étoilée sur Γ. L'auteur donne des conditions nécessaires et suffisantes pour que $f(z)$ soit convexe ou étoilée sur une courbe Γ, généralise un théorème d'Alexander [Ann. of Math. (2) **17**, 12–22 (1915)] en montrant que si $f(z)$ et $F(z)$ sont régulières sur Γ et si sur cette courbe $F(z)-w_0=hf'(z)dz/dt\neq 0$, h étant une constante, la convexité de $f(z)$ sur Γ entraîne l'étoilement de $F(z)$ sur Γ et inversement. Il étudie le cas où Γ est une ellipse ou une parallèle Γ_r à l'axe imaginaire, donne des inégalités précises pour les modules des coefficients d'une fonction $F(z)$ holomorphe pour $|z|<1$ lorsque $F(z)$ est étoilée sur les Γ_r et des inégalités corrélatives pour une fonction $f(z)$ de dérivée non nulle dans $|z|<1$, convexe sur les Γ_r. De ses résultats, l'auteur déduit que: si $p(z)$ est un polynome et Γ la frontière d'une région convexe contenant tous les zéros de $p(z)$, $p(z)$ est convexe sur Γ. Il étudie ensuite la convexité et l'étoilement sur des faisceaux de droites en relation avec la notion de fonction typiquement réelle de Rogosinski [Math. Z. **35**, 93–121 (1932)], puis les

bornes des coefficients de fonctions étoilées sur des familles de spirales logarithmiques ou de cercles. Il étudie enfin le cas de transformations non analytiques. *G. Valiron.*

Myrberg, P. J. Über automorphe Thetafunktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 111, 13 pp. (1952).

Let Γ be a fuchsian group in the unit circle. An automorphic theta function is defined as a function $g(z)$, regular in the unit circle, which under substitutions S of Γ is transformed in the following way (analogous to the elliptic theta functions): $g(S(z))=e^{\alpha_S u(z)+\beta_S}g(z)$, where $u(z)$ is regular in the unit circle. The author proves: If Γ is of genus zero, any automorphic function of Γ can be represented as a quotient of two automorphic theta functions, each of which is a product of primitive functions (i.e., having exactly one set of non-equivalent zeros). *H. D. Kloosterman* (Leiden).

Kufarev, P. P., and Fales, A. E. On an extremal problem for complementary regions. Doklady Akad. Nauk SSSR (N.S.) 81, 995–998 (1951). (Russian)

The authors consider the following problem, of which the special case $\alpha=\beta=1$ was treated earlier by one of the authors [Kufarev, same Doklady (N.S.) 73, 881–884 (1950); these Rev. 12, 401]. Let w_1 and w_2 be two fixed points in $|w|<1$, α and β two positive numbers, and Γ a Jordan arc which decomposes $|w|<1$ into two regions R_1 and R_2, such that $w_1 \in R_1$, $w_2 \in R_2$. It is required to find an arc Γ for which the functional $J(\Gamma; w_1, w_2)=|f_1'(0)|^\alpha |f_2'(0)|^\beta$ assumes its greatest value, where, for $k=1, 2$, the function $w=f_k(z)$, $f_k(0)=w_k$, $f_k'(0)>0$, maps $|z|<1$ conformally on R_k. *W. Seidel* (Princeton, N. J.).

Hersch, Joseph. Longueurs extrémales, mesure harmonique et distance hyperbolique. C. R. Acad. Sci. Paris 235, 569–571 (1952).

The author obtains estimates of the harmonic measure and hyperbolic distance of a plane configuration by relating these conformal invariants to the extremal length of Ahlfors and Beurling. Bounds on the extremal length are derived in terms of quasi-conformal mappings. *P. R. Garabedian.*

Piranian, G., Titus, C. J., and Young, G. S. Conformal mappings and Peano curves. Michigan Math. J. 1, 69–72 (1952).

A particularly simple example is given of a function $f(z)$, regular in $|z|<1$ and continuous in $|z|\leqq 1$, for which the values $f(e^{i\theta})$ fill a square. Other examples, of gap power series of this kind, were previously given by Salem and Zygmund [Duke Math. J. 12, 569–578 (1945); these Rev. 7, 378]; for these the unit circle is the natural boundary. This need not be the case for the new example.

W. W. Rogosinski (Newcastle-upon-Tyne).

Lohwater, A. J., and Piranian, G. Conformal mapping of a Jordan region whose boundary has positive two-dimensional measure. Michigan Math. J. 1, 1–4 (1952).

Construction d'une courbe de Jordan C, limitant un domaine borné R, sur laquelle il existe un ensemble E qui soit à la fois d'aire positive et de mesure harmonique nulle. Remarquons qu'il résulte des travaux de A. Denjoy [C. R. Acad. Sci. Paris 212, 1071–1074; 213, 15–17 (1941), p. 16; ces Rev. 5, 115, 116] que toute courbe d'aire positive possède cette propriété; car si $f(z)$ est univalente dans $|z|<1$, l'ensemble des points a de $|z|=1$ où $f'(z)=o((z-a)^{1/2})$ se transforme en un ensemble d'aire nulle, tandis que son complémentaire par rapport à $|z|=1$ est de mesure linéaire nulle. [Voir J. Ferrand, Ann. Sci. École Norm. Sup. (3) 59, 43–106 (1942), p. 70; ces Rev. 6, 207.] *J. Lelong.*

Piranian, George. Uniformly accessible Jordan curves through large sets of relative harmonic measure zero. Pacific J. Math. 2, 371–375 (1952).

Lohwater and Seidel [Duke Math. J. 15, 137–143 (1948); these Rev. 9, 420] have constructed a Jordan region whose boundary passes through a linear set of positive Lebesgue measure and of harmonic measure zero relative to the region. Also, Lohwater and Piranian [see the paper reviewed above] have constructed a Jordan region whose boundary passes through a set of positive two-dimensional Lebesgue measure and of harmonic measure zero. However, in both cases, the set in question on the boundary consists of points not finitely accessible from the interior. The author of the present paper shows that the last-mentioned fact is unessential, by constructing the regions of the respective properties whose boundaries are uniformly finitely accessible. A conjecture is stated concerning the radial limits of a function $f(z)=\sum a_n z^n$ with $\sum n|a_n|^2<\infty$. *Y. Komatu* (Tokyo).

Gergen, J. J., and Dressel, F. G. Mapping by p-regular functions. Duke Math. J. 18, 185–210 (1951).

Etude de la classe des fonctions $F(z)=u+iv$ (dites p-régulières) de la variable $z=x+iy$, qui satisfont à $pu_x'=v_y'$, $pu_y'=-v_x'$ dans $|z|<a$, $p(x, y)$ étant une fonction réelle positive donnée pourvue de dérivées bornées.

La première partie est consacrée à l'étude des fonctions continues $w(z)$ qui satisfont à $|w_*|<M|w|$, w_* étant la fonction définie par

$$\int_c w dz = i \int\int_\sigma w_* dS$$

où c est un rectangle quelconque de côtes parallèles aux axes et σ son intérieur. Diverses formules sont établies, qui permettent de rattacher ces fonctions aux fonctions analytiques, et d'étudier leurs zéros.

La deuxième partie constitue une extension, aux fonctions p-harmoniques u, v, de certaines propriétés des fonctions harmoniques. Ces fonctions réalisent en particulier le minimum de l'intégrale $\int\int p|\nabla w|^2 dS$ parmi toutes les fonctions w prenant mêmes valeurs frontières.

Les méthodes de Douglas et Courant permettent alors (troisième partie) d'étendre, aux transformations définies par les fonctions p-régulières, le théorème de Riemann sur la representation conforme (existence d'une représentation de $|z|<a$ sur un domaine limité par une courbe rectifiable, avec correspondance de trois couples frontières donnés).

Les auteurs font remarquer que les transformations considérées ne sont pas quasi-conformes au sens habituel, mais seulement au sens, plus général, de Lavrentieff [Mat. Sbornik N.S. 21(63), 285–320 (1947); Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 513–554 (1948); ces Rev. 10, 290; 11, 650]. Leur résultat principal est sensiblement contenu dans celui obtenu par Lavrentieff pendant la révision de leur mémoire, mais les méthodes sont différentes. *J. Lelong.*

Gergen, J. J., and Dressel, F. G. Uniqueness for p-regular mapping. Duke Math. J. 19, 435–444 (1952).

Unicité de la représentation dont l'existence a été établie dans l'article analysé ci-dessus, et de représentations analogues. Mêmes méthodes et mêmes notations.

J. Lelong (Lille).

Sagawa, Akira. Über die Ausnahmegebiete. Math. Japonicae **2**, 146–148 (1952).

Elaboration of a condition of Dinghas [Math. Z. **44**, 568–572 (1938)] for the Nevanlinna defect of a region D to vanish. *P. R. Garabedian* (Stanford, Calif.).

***Nevanlinna, Rolf. Über den Gauss-Bonnetschen Satz.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 175–178. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

Identical with Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **100** (1951); these Rev. **13**, 644.

Parreau, M. Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Ann. Inst. Fourier Grenoble **3** (1951), 103–197 (1952).

The present paper is a comprehensive study of subharmonic, harmonic, holomorphic, and meromorphic functions with bounded mean of order α on open Riemann surfaces.

At the outset, the Perron-Brelot method is used to solve the Dirichlet problem on compact Riemann domains; harmonic functions with prescribed singularities are then constructed on arbitrary Riemann surfaces by means of the generalized alternating method [Sario, C. R. Acad. Sci. Paris **229**, 1293–1295 (1949); these Rev. **11**, 342]. The author also shows that main theorems in the theory of Newtonian potentials can be extended to Green's potentials.

To define mean values, consider a compact subdomain G of an open Riemann surface S. Let the boundary of G consist of two finite sets C_0 and C_1 of disjoint analytic curves. Denote by x the harmonic function in G with $x=0$ on C_0, $x=\text{const.}=\Lambda$ on C_1, $\int d\tilde{x}=1$ along C_0, where $\tilde{x}$ is the harmonic conjugate of x. The mean value of order α of a harmonic function u on $\bar{G}$ is defined by

$$m_\alpha(\lambda)=m_\alpha(\lambda, u; G)=\left(\int_{x=\lambda}|u|^\alpha d\tilde{x}\right)^{1/\alpha},$$

where $\alpha\geqq 1$ and $0\leqq\lambda\leqq\Lambda$. For every u, $m_\alpha(\lambda)$ is a convex function of λ; for functions u with $u=0$ on C_0, $\int|d\tilde{u}|=1$ along C_0, the relation $m_\alpha(\lambda)\geqq\lambda$ holds. This is an extension of a corresponding theorem, for $\alpha=2$, of Nevanlinna [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **57** (1949); these Rev. **11**, 516]. The author defines a harmonic function u to be of bounded mean of order α on S if $m_\alpha(\lambda)$ is uniformly bounded in $\{S_1-S_0\}$ for any fixed compact $S_0\subset S$ and all compact $S_1\supset S_0$. The class of these functions is denoted by HM_α. A necessary and sufficient condition for $u\in HM_\alpha$ is that the subharmonic function $|u|^\alpha$ is dominated by a harmonic function on S. The existence of a function $u\in HM_\alpha$ on S depends merely on the ideal boundary of S. For $\alpha>1$, the class O_{HM_α} of Riemann surfaces without nonconstant functions HM_α coincides with the class O_{HB} of surfaces without (nontrivial) harmonic bounded functions. For $\alpha=1$, the question remains open.

Let H_0M_α be the class of functions $u\in HM_\alpha$ in $S-S_0$ with $u=0$ on the boundary C_0 of S_0. The author shows that the following theorems of the reviewer [Congr. Scand. Math., Trondheim, 1949, pp. 229–238 (in press); C. R. Acad. Sci. Paris **230**, 42–44 (1950); these Rev. **11**, 342] continue to hold for $O_{H_0M_\alpha}$ with $\alpha\geqq 1$: (1) $O_{H_0M_1}=O_{H_0B}=O_{H_0D}=O_G$, where B and D stand for "bounded" and "with a finite Dirichlet integral" respectively, and O_G is the class of parabolic surfaces. (2) These relative classes are included in the corresponding absolute classes O_{HM_α}, O_{HB}, O_{HD}. (3) For surfaces with finite genus, these seven classes coincide. (4) A necessary and sufficient condition for $S\in O_{HB}$ or O_{HD} is that every $u\in H_0B$ or H_0D in $S-S_0$ with $\int d\tilde{u}=0$ vanishes identically.

The author then proves that every function $u\in HM_\alpha$ can be uniquely represented in the form $u(p)=\int_\Gamma K(p, s)d\mu(s)$ where $K(p, s)=g(p, q)/g(a, q)$ defines, in the sense of R. S. Martin [Trans. Amer. Math. Soc. **49**, 137–172 (1941); these Rev. **2**, 292], boundary elements when the pole q of the Green's functions g tends to the point s of the ideal boundary Γ, and μ is a canonical measure on the Martin boundary.

Let u be a positive harmonic function on S, and let u_B be the "upper envelope" of bounded harmonic functions dominated by u on S. The author calls u quasi-bounded or singular if $u_B=u$ or $u_B=0$ respectively. For $\alpha>1$, every u of the class HM_α is quasi-bounded. Every positive harmonic function on S can be uniquely expressed as the sum of a quasi-bounded and a singular harmonic function. The function which minimizes the norm among $u\in HM_\alpha$, $u(q)=1$ (q fixed) is bounded. For $\alpha=2$, the minimizing function is a Bergman kernel function.

Several of these results remain valid if the operation $|u|^\alpha$ in $m_\alpha(\lambda)$ is replaced by a more general operation $\Phi(|u|)$ where $\Phi(t)$ is a convex function of $t>0$ with $\Phi(0)=0$.

For holomorphic functions, the mean is defined as for harmonic functions. The convexity theorem remains valid. If the real part of a holomorphic function f belongs to HM_α, then f belongs to the corresponding class AM_α of analytic functions. A meromorphic function f is defined to be of class AM_0 if $\log^+|f|$ is dominated by a superharmonic function on S. This is identical to the condition that $\log|f|$ is a difference of two positive superharmonic functions. The author introduces a characteristic function $T(G, f)$ of f whose variable is a domain G. A necessary and sufficient condition for $f\in AM_0$ is that $T(G, f)$ remain bounded. For meromorphic functions with bounded characteristic, Frostman's and Valiron-Ahlfors-Nevanlinna's theorems on defect values continue to hold. *L. Sario* (Cambridge, Mass.).

Mori, Akira. A remark on the prolongation of Riemann surfaces of finite genus. J. Math. Soc. Japan **4**, 27–30 (1952).

Let O_{AD} be the class of Riemann surfaces F not admitting single-valued nonconstant analytic functions with a finite Dirichlet integral. If F is of finite genus, it can be mapped conformally into a closed surface F^*, the boundary Γ of F appearing as a point set interior to a simply connected domain $K\subset F^*$. For a planar F, it is known [Sario, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); these Rev. **10**, 365] that every AD-function in $K-\Gamma$ can be continued analytically over Γ if and only if $F\in O_{AD}$. The author shows that this remains true for a positive genus, an extension found also by the reviewer [Congr. Scand. Math., Trondheim, 1949, pp. 229–238 (in press)]. As an application, the author shows that the mapping of F into F^* is uniquely determined up to a conformal self-mapping of F^* if and only if $F\in O_{AD}$. The sufficiency of this condition was established by the reviewer [loc. cit. 1949], the necessity in the planar case by Ahlfors and Beurling [Acta Math. **83**, 101–129 (1950); these Rev. **12**, 171], the extension for positive genus being announced by the reviewer [Proc. Internat. Congr. Math., Cambridge, Mass., 1950, vol. I, pp. 398–399, Amer. Math. Soc., Providence, R. I., 1952]. *L. Sario.*

Mori, Akira. A remark on the class O_{HD} of Riemann surfaces. Kôdai Math. Sem. Rep. **1952**, 57–58 (1952).

Let G_1, G_2 be disjoint subregions of an open Riemann surface F, with compact or noncompact relative boundaries C_1, C_2. Denote by HD the class of single-valued nonconstant harmonic functions with a finite Dirichlet integral, and by O_{HD} the class of Riemann surfaces on which there is no HD-function. On G_1, G_2, let H_0D be the class of HD-functions which vanish on C_1, C_2. It is known that F non-$\in O_{HD}$ if there is an H_0D-function on G_1 and G_2 [Bader and Parreau, C. R. Acad. Sci. Paris **232**, 138–139 (1951); these Rev. **12**, 603]. For compact C_1, C_2, the latter condition is equivalent to the existence of an H_0B-function (B=bounded) on G_1 and G_2 [Sario, Congr. Scand. Math., Trondheim, 1949, pp. 229–238 (in press)]. The author shows that, for noncompact C_1, C_2, it suffices that there be an H_0B-function in G_1 and an H_0D-function in G_2. *L. Sario* (Cambridge, Mass.).

Cartan, Henri. Sur une extension d'un théorème de Radó. Math. Ann. **125**, 49–50 (1952).

Verf. gibt einen einfachen Beweis einer Verallgemeinerung eines Radó'schen Satzes, die sich in einer kürzlich erschienenen Arbeit von H. Behnke und K. Stein findet [Math. Ann. **124**, 1–16 (1951); diese Rev. **13**, 644]. Die von H. Cartan angegebene Fassung lautet: $\mathfrak{G}$ sei eine analytisch komplexe Mannigfaltigkeit, ferner g eine in $\mathfrak{G}$ definierte und dort überall stetige Funktion, die zudem in jedem Punkt von $\mathfrak{G}$ wo $g\neq 0$ analytisch sei; dann ist g eine in ganz $\mathfrak{G}$ analytische Funktion. Der neue Beweis benutzt elementare Überlegungen der Theorie der subharmonischen Funktionen einer komplexen Veränderlichen. *P. Thullen* (Genf).

Oka, Kiyoshi. Sur les fonctions analytiques de plusieurs variables. VIII. Lemme fondamental (suite). J. Math. Soc. Japan **3**, 259–278 (1951).

Der Verfasser hat zum Beweise seiner zahlreichen Ergebnisse [J. Sci. Hiroshima Univ. Ser. A. **6**, 245–255 (1936); **7**, 115–130 (1937); **9**, 7–19 (1939); Jap. J. Math. **17**, 517–521, 523–531 (1941); Tôhoku Math. J. **49**, 15–52 (1942); Bull. Soc. Math. France **78**, 1–27 (1950); J. Math. Soc. Japan **3**, 204–214 (1951); diese Rev. **3**, 85; **7**, 290; **12**, 18; **13**, 454] insbesondere ein neues fruchtbares Prinzip eingeführt, das seitdem weithin mit Erfolg benutzt wird. Einem beliebigen Polyedergebiet $\mathfrak{P}$ des R^{2n} wird ein Zylinderbereich $\mathfrak{Z}$ des R^{2m}, $m>n$, so zugeordnet, dass die in $\mathfrak{P}$ regulären Funktionen durch Einsetzen von $m-n$ analytischen Gleichungen entstehen. So wird die behandelte Fragestellung in $\mathfrak{P}$ in Verbindung mit der entsprechenden Fragestellung in einem Gebiete von zwar höherer Dimension aber im allgemeinen einfacher topologischer Struktur gebracht. Der für die Anwendung dieses Prinzips wesentliche Satz lautet: Ist $\mathfrak{P}$ gegeben durch

$$\{z_\nu \in \overline{\mathfrak{G}}_\nu,\ f_\mu(z_1, \cdots, z_n) \in \overline{\mathfrak{G}}_\mu^*,\ \nu=1, \cdots, n;\ \mu=1, \cdots, m\},$$

wo die $\overline{\mathfrak{G}}_\nu$, $\overline{\mathfrak{G}}_\mu^*$ abgeschlossene ebene Gebiete und die $f_\mu(z_1, \cdots, z_n)$ reguläre Funktionen in einem $\mathfrak{P}$ umfassenden Gebiete $\mathfrak{G}$ des R^{2n} sind, so werde $\mathfrak{P}$ die $2n$-dimensionale analytische Mannigfaltigkeit

$$\mathfrak{M}: \quad \{w_\mu = f_\mu(z_1, \cdots, z_n),\ (z_1, \cdots, z_n) \in \mathfrak{P}\}$$

im abgeschlossenen Zylindergebiete

$$\overline{\mathfrak{Z}}: \quad \{z_\nu \in \overline{\mathfrak{G}}_\nu,\ w_\mu \in \overline{\mathfrak{G}}_\mu^*\}$$

des Raumes $R^{2(n+m)}$ der komplexen Veränderlichen $z_1, \cdots, z_n$, $w_1, \cdots, w_m$ zugeordnet. $g(z_1, \cdots, z_n)$ sei eine in $\mathfrak{P}$ reguläre Funktion; sie lässt sich als Funktion auf $\mathfrak{M}$ auffassen. Dann existiert eine in $\overline{\mathfrak{Z}}$ reguläre Funktion $\mathfrak{G}(z_1, \cdots, z_n, w_1, \cdots, w_m)$, die auf $\mathfrak{M}$ mit g übereinstimmt. [Siehe Verf., J. Sci. Hiroshima Univ. Ser. A. **6**, 245–255 (1936).] Die Anwendung dieses Satzes machte bisher Schwierigkeiten, wenn $\mathfrak{P}$ innere Verzweigungspunkte hatte. Die überwindet Verf. nun in der vorliegenden Arbeit. $\mathfrak{P}$ sei dem R^{2n} nichtschlicht überlagert. Es werden ebenso Verzweigungspunkte als innere Punkte von $\mathfrak{P}$ zugelassen. Nun wird eine Hilfsfunktion u_0 eingeführt. Das ist erforderlich, da eine wörtliche Übertragung des Satzes für diesen Fall nicht möglich ist, denn eine reguläre Funktion von n Veränderlichen auf einer analytischen Fläche im $R^{2(n+p)}$, $p>0$, ist in der Umgebung eines nicht gewöhnlichen Punktes dieser Fläche im allgemeinen nicht einmal lokal zu einer regulären Funktion im $R^{2(n+p)}$ fortsetzbar. Verf. zeigt nun jedoch: Sind $\mathfrak{P}$ und $\mathfrak{G}$ Teilbereiche des Existenzgebietes der regulär algebroiden Funktion f_1 und werden die $\mathfrak{G}_\nu$, $\mathfrak{G}_\mu^*$ als einfach-zusammenhängend vorausgesetzt, so existiert in $\mathfrak{P}$ eine regulär algebroide Funktion $u_0\not\equiv 0$ und endlich viele in $\overline{\mathfrak{Z}}$ reguläre Funktionen $\phi_\rho(z_1, \cdots, z_n, w_1, \cdots, w_m)$, $\rho=1, \cdots, r$, derart dass für jede in $\mathfrak{P}$ regulär algebroide Funktion g stets $u_0 \cdot g$ (als Funktion auf $\mathfrak{M}$ aufgefasst) zu einer regulären Funktion $\mathfrak{G}(z_1, \cdots, z_n, w_1, \cdots, w_m)$ in $\overline{\mathfrak{Z}}$ fortsetzbar ist, und dass $\mathfrak{G}\equiv 0 \bmod (\phi_1, \cdots, \phi_r)$ in $\overline{\mathfrak{Z}}$ gilt. Abschliessend gibt Verf. Anwendungen seines Satzes, die den Cousinschen Problemkreis, die Approximation regulärer Funktionen in analytischen Polyedern sowie die Frage nach der Existenz lokaler Pseudobasen zu Idealen regulärer Funktionen mit unbestimmten Gebieten betreffen. [Siehe H. Cartan, Bull. Soc. Math. France **78**, 29–64 (1950); diese Rev. **12**, 172.]

H. Behnke (Münster).

Hitotumatu, Sin. A note on the maximal ideals of analytic functions. Kôdai Math. Sem. Rep. **1952**, 51–53 (1952).

An ideal of analytic functions on a domain of holomorphy D is called closed if it is closed in the topology of uniform convergence on compact subsets of D. The author proves that a closed maximal ideal M on a domain of holomorphy is the ideal of functions vanishing at some point $(a_1, \cdots, a_n)$ in D and $M=(z_1-a_1, \cdots, z_n-a_n)$. He further proves that a maximal ideal with a countable generating set is closed. He conjectures that the countability assumption is unnecessary. This is not so. Let D be a domain in one variable and let X be an infinite discrete subset of D. Let U be a non-trivial ultrafilter on X. Then the ideal of those functions on D which vanish on an element of U is a non-closed maximal ideal. The author also asks whether a closed ideal on a domain of holomorphy has a finite basis. It can be shown by example that it need not. *N. T. Hamilton.*

Tibiletti, Cesarina. Alcune estensioni di un teorema di Noether. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 412–428 (1951).

Extension du théorème de Noether au champ analytique. Une fonction f analytique de deux variables, régulière en tout point à distance finie peut se mettre sous la forme $A\varphi+B\psi$, φ et ψ algébriques, A et B analytiques régulières en tout point à distance finie si dans le voisinage de tout point commun à φ et ψ la f peut s'écrire $A'\varphi+B'\psi$, les A' et B' étant des séries de puissance. Traduction de la propriété en termes d'idéaux. *B. d'Orgeval* (Alger).

Ganin, M. P. Boundary value problems for polyanalytic functions. Doklady Akad. Nauk SSSR (N.S.) **80**, 313–316 (1951). (Russian)

By a polyanalytic function $F(z, \bar{z})$ the author means a function representable in the form

$$F=\varphi_0(z)+|z|^2\varphi_1(z)+\cdots+|z|^{2n-2}\varphi_{n-1}(z),$$

the φ's being holomorphic. He wants to find, in an $(m+1)$-times connected domain with a sufficiently smooth boundary C, a polyanalytic function $F=U+iV$ satisfying one of the following conditions of the form:

$$(\alpha)\qquad a_k\Delta^k U+b_k\Delta^k V=g_k;$$

$$(\beta)\qquad a_k^*(\partial^k U/\partial n)+b_k^*(\partial\Delta^k V/\partial n)=g_k^*;$$

$$(\gamma)\qquad a_k(\partial^k U/\partial n^k)+b_k(\partial^k V/\partial n^k)=g_k;$$

$$(\delta)\qquad a_k(\partial^{n-1}U/\partial x^{n-k}\partial y^{k-1})+b_k(\partial^{n-1}V/\partial x^{n-k}\partial y^{k-1})=g_k.$$

Here a_k, b_k, a_k^*, b_k^*, g_k, g_k^* are given sufficiently smooth functions defined on C. Problem I involves condition (α) for $k=0, 1, \cdots, n-1$, Problem II conditions (α) and (β) for $k=0, 1, \cdots, [(n-1)/2]$, Problem III condition (γ) for $k=0, \cdots, n-1$, Problem IV condition (δ) for $k=1, 2, \cdots, n$. Problem I is reduced to the Hilbert-Riemann problem, Problems II–IV to systems of Fredholm integral equations.

L. Bers (New York, N. Y.).

Theory of Series

Zmorovič, V. A. On a criterion of N. I. Lobačevskiĭ for the convergence of positive numerical series and a generalization of this criterion. Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 162–170 (1952). (Russian)

The author gives an elementary discussion and simple modifications of the following theorem published by Lobačevskiĭ [Učenye Zapiski Imp. Kazan. Univ. **1834**, Kn. II, 167–226]. Let $\sum_{n=1}^{\infty}u_n$ be a series of positive terms for which $u_{n+1}<u_n$ and $u_n\to 0$. Let k be a positive integer. Then $\sum u_n$ converges if and only if $\sum p_n 2^{-n}$ converges when the integers p_n are determined by the conditions $U_{p_n+1}<2^{-n}u_k\leqq U_{p_n}$, $n=1, 2, 3, \cdots$.

R. P. Agnew (Ithaca, N. Y.).

Žitomirskiĭ, Ya. I. On the convergence of certain numerical series. Uspehi Matem. Nauk (N.S.) **7**, no. 4(50), 153–156 (1952). (Russian)

Simple estimates related to the integral test for convergence of series are used to prove the following theorem. Let $g(x)$ be a positive continuous monotone decreasing function for which $g(x)\to 0$ as $x\to\infty$. Let $\sum$ denote the series $g(1)-g(2)+g(3)-g(4)+\cdots$. Let $\sum'$ denote a rearrangement of $\sum$ such that the first N terms of $\sum'$ consist of the first $p(N)$ positive terms of $\sum$ and the first $q(N)$ negative terms of $\sum$. Then $\sum'$ converges if and only if $\lim_{N\to\infty}\int_{2q(N)}^{2p(N)}g(x)dx$ exists.

R. P. Agnew (Ithaca, N. Y.).

Maschler, Michaël. Sur une transformation généralisée de série en série. C. R. Acad. Sci. Paris **235**, 769–771 (1952).

The author generalizes known theorems which characterize the class of matrices a_{nk} of series-to-series transformations $v_n=\sum_{k=0}^{\infty}a_{nk}u_k$ such that $\sum_{n=0}^{\infty}v_n=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{nk}u_k$ converges whenever $\sum u_n$ converges, and the subclass of these matrices for which the iterated series converges to s whenever $\sum u_n$ converges to s. The generalization involves matrices with elements having s subscripts where s is an integer greater than 1. The case $s=3$ involves the iterated series $\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{n,j,k}u_k$.

R. P. Agnew.

Peyerimhoff, Alexander. Konvergenzfaktoren beim Euler-Knoppschen Limitierungsverfahren. Math. Z. **55**, 288–291 (1952).

If $\sum s_n\epsilon_n$ converges whenever $\{s_n\}$ is convergent-E^k (or (E, q), $q=2^k$), i.e., by the Euler-Knopp process, it is necessary that $\epsilon_n=O\{(2q-1)^{-n}\}$. On the other hand it is sufficient that $\epsilon_n=\alpha_n/(2q-1)^n$, where $\sum|\alpha_n|<\infty$, or that $\epsilon_n=(2q-1)^{-n}$.

L. S. Bosanquet (London).

Karamata, J. Ein Satz über die Abschnitte einer Potenzreihe. Math. Z. **56**, 219–222 (1952).

A series $\sum u_n$ is said to be summable A_N $[N=N(x)]$ if $\sum_{r\leq N}u_r x^r$ tends to a limit as $x\to 1-0$. If $N(x)=\log t/\log x$ and $\frac{1}{2}<t<1$, summability A_N is equivalent to convergence, and this result is substantially best possible. On the other hand, Abel summability, together with the Tauberian condition $nu_n=O(1)$ implies summability A_N for $0<t<1$. The two results together provide a simple and elementary proof of Littlewood's Tauberian theorem for Abel summability.

H. R. Pitt (Nottingham).

Erdös, P. On a Tauberian theorem for Euler summability. Acad. Serbe Sci. Publ. Inst. Math. **4**, 51–56 (1952).

Meyer-König [Math. Z. **45**, 479–494 (1939); these Rev. **1**, 51] has proved that if $a_n=0$ except if $n=n_i$, where $n_{i+1}-n_i>An_i^{1/2}$, $A>0$, if $|a_n|\leqq n^\alpha$ and if $\sum a_n$ is $E_{1/2}$ summable, that is, if $\sum a_n'$ converges with $a_n'=2^{-(n+1)}\sum_{k=0}^{n}\binom{n}{k}a_k$, then $\sum a_n$ converges. Pitt [Proc. London Math. Soc. (2) **44**, 243–288 (1938)] has this even with $|a_n|\leqq(1+\epsilon)^n$ for each $\epsilon>0$. Here it is shown that for A sufficiently large the theorem is true without any restriction on the magnitude of $|a_n|$. Meyer-König's conjecture that this is true for any $A>0$ remains open.

G. G. Lorentz (Toronto, Ont.).

Rajagopal, C. T. Note on some Tauberian theorems of O. Szász. Pacific J. Math. **2**, 377–384 (1952).

Let $\varphi(x)$ satisfy, for $x\geqq 0$, a set of 5 specified conditions which are satisfied by the functions e^{-x}, $(1-x)^{-r}$ when $r>0$, and $\varphi_1(t)=x/(e^x-1)$ for $x>0$ while $\varphi_1(0)=1$. Let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ and $\lambda_n\to\infty$. Let $\sum a_n$ be a real series for which $\sum a_n\varphi(\lambda_n t)$ converges when $t>0$ to a function $\Phi(t)$ such that $\Phi(t)\to s$ as $t\to 0+$. Several Tauberian theorems such as the following are proved. If $\lambda_{n+1}/\lambda_n\to 1$ and, for some constant $p>1$, $\sum_{k=1}^{n}(|a_k|-a_k)^p\lambda_k^p(\lambda_k-\lambda_{k-1})^{1-p}=O(\lambda_n)$, then $\sum a_n$ converges to s. The theorems generalize theorems of O. Szász [same J. **1**, 117–125 (1951); these Rev. **13**, 227] for which $\varphi(x)=e^{-x}$ and the function $\Phi(t)$ is the Abel transform of $\sum u_n$.

R. P. Agnew (Ithaca, N. Y.).

Wielandt, Helmut. Zur Umkehrung des Abelschen Stetigkeitssatzes. Math. Z. **56**, 206–207 (1952).

This paper gives, on two pages, a complete proof of the unilateral Hardy-Littlewood Tauberian theorem for the Abel power series transformation. The theorem says that if $na_n\leqq C$ and $\sum a_n x^n\to s$ as $x\to 1-$, then $\sum a_n$ converges to s. The proof is obtained by an ingenious modification of the Karamata method [Math. Z. **32**, 319–320 (1930)].

R. P. Agnew (Ithaca, N. Y.).

Meyer-König, W. Das Taylorsche Verfahren zur Limitierung von Funktionen. Math. Z. **56**, 179–205 (1952).

A series $\sum a_n$ with partial sums s_n is said to be evaluable to s by the Taylor method T_α if $t_n\to s$ where

$$t_n=(1-\alpha)^{n+1}\sum_{k=n}^{\infty}\binom{k}{n}\alpha^{k-n}s_k.$$

This definition is motivated by the Weierstrass method of analytic continuation by use of Taylor series. Similar motivation leads to definitions whereby the integral $\int_0^\infty a(x)dx$ with partial integrals $s(x)=\int_0^x a(t)dt$ may be evaluable. The integral and the function $s(x)$ are said to be evaluable (or to be limitable) to s by the Taylor method T_α' if

$$s(0)+\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\int_0^\infty t^n e^{-\alpha t}ds(t)=s.$$

Interest in this method can also be generated by the purely formal manipulation

$$\begin{aligned}s(\infty)&=s(0)+\int_0^\infty ds(t)\\&=s(0)+\int_0^\infty e^{-\alpha t}\sum_{n=0}^{\infty}\frac{\alpha^n t^n}{n!}ds(t)\\&=s(0)+\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\int_0^\infty t^n e^{-\alpha t}ds(t).\end{aligned}$$

Problems involving regularity of the methods T_α', and their effectiveness in evaluating Laplace integrals, are solved. The methods T_α' and closely related methods are compared with each other and with methods of Cesàro, Abel, and Borel.

R. P. Agnew (Ithaca, N. Y.).

Cowling, V. F., and Piranian, G. On the summability of ordinary Dirichlet series by Taylor methods. Michigan Math. J. 1, 73–78 (1952).

Die Reihe $\sum A_n$ heisst (absolut) T_α-summierbar ($0<\alpha<1$), wenn die Reihe $\sum A_n'$ mit $A_n'=(1-\alpha)^n\sum_{k=n}^{\infty}\binom{k}{n}\alpha^{k-n}A_k$ (absolut) konvergiert. Das so definierte Taylorsche Summierungsverfahren wurde von Hardy und Littlewood [Rend. Circ. Mat. Palermo **41**, 36–53 (1916)] eingeführt und von R. Wais [Dissertation, Tübingen, 1935] und W. Meyer-König [Math. Z. **52**, 257–304 (1949); diese Rev. **11**, 242] weiteruntersucht. Verff. wenden hier das T_α-Verfahren zur Summierung gewöhnlicher Dirichletscher Reihen $\sum a_n(n+1)^{-s}$ an und beweisen die folgenden Sätze. Es sei $\limsup|a_n|^{1/n}<1/\alpha$, $0<\alpha<1/3$. Satz 1: Existiert die $T_\alpha-\sum a_n(n+1)^{-s_0}$, so ist $\sum a_n(n+1)^{-s}$ für $\Re(s)>\Re(s_0)+1$ $|T_\alpha|$-summierbar. Satz 2: Existiert $|T_\alpha|-\sum a_n(n+1)^{-s}$ für $s=s_0$, so auch für $\Re(s)>\Re(s_0)$. Ganz entsprechende Untersuchungen bei der Anwendung des Euler-Knoppschen E_p-Verfahrens auf gewöhnliche Dirichletsche Reihen hat N. Obrechkoff [S.-B. Preuss. Akad. Wiss. **1938**, 267–300] angestellt. Dem Ref. erscheint eine Ausdehnung der beiden Sätze auf den Fall $0<\alpha<\frac{1}{2}$ durch geringe Modifikation der Beweisführung möglich.

D. Gaier (Stuttgart).

Zonneveld, J. A., and Berghuis, J. Asymptotic expansions connected with truncated series of exponential and Bessel type. Computation Department, Mathematical Centre, Amsterdam. Rep. **MR 8**, i+13 pp. (1952).

Let $\phi(n,w)(nw)^n/n!$ be the remainder R_n after n terms of the Maclaurin series for e^{nw}. Asymptotic expansions of $\phi(n,w)$ are obtained which include previous results by several authors following a conjecture by Ramanujan [Collected papers, Cambridge, 1927, p. XXVI, no. VII]. These expansions are obtained from a representation of ϕ as a Laplace transform of a function Φ which, in turn, is found by making an elementary substitution in the Lagrange integral form for the remainder R_n and deforming the new path of integration C into the real line from zero to infinity. Two cases arise, leading to two different kinds of asymptotic expansions, according to whether or not C encounters a branch point of Φ during this deformation. Similar results are obtained by truncating the series $\sum_n x^n/[n!(n+k)!]$, a series which is chosen because it is closely related to but simpler than that for the Bessel function $J_k(x)$. The associated asymptotic expansions for this series are used to estimate the number of real zeros of the polynomial which is the first m terms of the series.

P. W. Ketchum.

Scott, W. T. The reciprocal of a continued fraction. Proc. Amer. Math. Soc. **3**, 722–726 (1952).

Conditions on the coefficients a_p are found such that the correspondence

$$F(z)\sim 1+\cfrac{a_1}{z^{\beta_1}+1}+\cfrac{a_2}{1}+\cdots+\cfrac{a_{2n-1}}{z^{\beta_{2n-1}}+1}+\cfrac{a_{2n}}{1}+\cdots$$

implies the correspondence of the reciprocal

$$\frac{1}{F(z)}\sim 1+\cfrac{a_1^*}{z^{\beta_1}+1}+\cfrac{a_2^*}{1}+\cdots+\cfrac{a_{2n-1}^*}{z^{\beta_{2n-1}}+1}+\cfrac{a_{2n}^*}{1}+\cdots,$$

where $F(z)=1+\sum_1^n c_p/z^p$. (Formulas (10) derived here for the establishment of this correspondence are a special case of those found in Theorem 6.5, pp. 106–107 of E. Frank [Amer. J. Math. **68**, 89–108 (1946); these Rev. **7**, 434] with $\alpha_1=\alpha_2=\beta_1=\beta_2$, $\alpha_{2p-3}=\alpha_{2p-2}=\beta_{2p-3}=\beta_{2p-2}=0$, $p=3,\cdots,n$, $\alpha_{2p-1}=\alpha_{2p}=\beta_{2p-1}=\beta_{2p}$, $\cdots$.)

E. Frank.

Fourier Series and Generalizations, Integral Transforms

Calderon, A. P., and Klein, G. On an extremum problem concerning trigonometrical polynomials. Studia Math. **12**, 166–169 (1951).

Si dimostra il seguente teorema, che ne generalizza uno di P. Erdös [Acta Litt. Sci. Szeged **9**, 113–115 (1939)]: Sia $\varphi(x)$ una funzione non negative definita per x non negativa, tale che $[\varphi(x)-\varphi(0)]/x$ sia non decrescente rispetto ad x ($\geqq 0$); sia Σ l'insieme dei polinomi trigonometrici d'ordine n, limitati in modulo da 1; allora l'integrale $\int_0^{2\pi}\varphi[|S'(x)|]dx$ ($S(x)\in\Sigma$) assume il suo massimo valore in Σ per

$$S(x)\equiv\cos(nx+\alpha);$$

nel caso che $\varphi(x)$ non sia costante, $\cos(nx+\alpha)$ è l'unico elemento massimante di Σ.

F. Bertolini (Roma).

Tomić, M. Einige Sätze über die Positivität der trigonometrischen Polynome. Acad. Serbe Sci. Publ. Inst. Math. **4**, 145–156 (1952).

The author first gives a new proof of the inequality $\sum_{\nu=1}^{n}\nu^{-1}\sin\nu\theta>0$ for $0<\theta<\pi$. He then proves a general theorem which includes it, namely that

$$S_n(\theta)=\sum_{\nu=1}^{n}c_\nu\sin\nu\theta\geqq 0$$

for $0\leqq\theta\leqq\pi$, and all sections of $S_n(\theta)$ have the same property, provided that the finite sequence $\{c_k\}_1^n$ is twice monotonic and the sequence $\{C_m\}_1^n$ is positive, where

$$C_m=\sum_{\nu=1}^{m}(-1)^{\nu-1}\nu c_\nu.$$

This overlaps Fejér's theorem [Trans. Amer. Math. Soc. **39**, 18–59 (1936)] that $S_n(\theta)\geqq 0$ if $\{c_k\}_1^{n+1}$ is twice monotonic ($c_{n+1}=0$). The proof depends on results of Fejér [loc. cit.

and Monatsh. Math. Phys. **35**, 305–344 (1928)]. By a geometric method which he has used elsewhere [same Publ. **2**, 146–156 (1948); **3**, 243–258 (1950); these Rev. **10**, 452; **12**, 813], the author also proves that $S_n(\theta)\geqq 0$ in the smaller interval $(0, \pi/2)$ under weaker conditions.

R. P. Boas, Jr. (Evanston, Ill.).

Tomić, M. Sur les sommes trigonométriques à coefficients monotones. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 13–52 (1952). (Serbo-Croatian. French summary)

This paper contains some of the results which the author has discussed elsewhere [cf. the preceding review and references given there; also Karamata and Tomić, Acad. Serbe Sci. Publ. Inst. Math. **2**, 157–175 (1948); **3**, 207–218 (1950); these Rev. **10**, 452; **12**, 482], and some additional material. Among the new results are the following. New proofs of some theorems on the location of the zeros of trigonometric series of special form, and on the positivity of various trigonometric polynomials. If $\{a_\nu\}$ is quadruply monotonic,

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu\theta \leqq \tfrac{1}{2}a_1 \cot \tfrac{1}{2}\theta.$$

Let $R_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$, $r_n(z)=z^{-n}R_n(z)$; if $\{c_\nu\}$ is triply monotonic, then $\{|r_\nu(z)-r_{\nu+1}(z)|\}$ is a decreasing sequence. Let $c_\nu=\rho_\nu e^{\Omega_\nu}$; if $\{\rho_\nu\}$ decreases and $\{\lambda_\nu\}$ is triply monotonic, then $\sum_{\nu=0}^{n} c_\nu e^{i\nu\theta}$ converges uniformly in $0\leqq\theta\leqq\pi-\lim(\lambda_\nu-\lambda_{\nu+1})$, and has no zeros there. [The theorem on boundedness or convergence of a sine series with monotonic coefficients in §2.3 is not new; cf. Hardy and Rogosinski, Fourier series, Cambridge, 1944, p. 35; these Rev. **5**, 261.]

R. P. Boas, Jr. (Evanston, Ill.).

Yano, Shigeki. Note on Fourier analysis. XXXI. Cesàro summability of Fourier series. Pacific J. Math. **2**, 419–429 (1952).

The author proves the following theorem: Let

$$\sum_{1}^{\infty}(a_n \cos nx+b_n \sin nx)$$

be the Fourier series of a function $f\in L$. Then both series $\sum(a_n \cos nx+b_n \sin nx)n^{-\alpha}$ and $\sum(a_n \sin nx-b_n \cos nx)n^{-\alpha}$ $(0<\alpha<1)$ are summable $(C, -\alpha)$ almost everywhere. The author quotes a remark of A. Zygmund according to which this theorem can also be obtained directly from known results. The analogous theorem for double Fourier series is also proved. Finally, the author proves that the set E of points where the series $\sum(a_n \cos nx+b_n \sin nx)n^{-\alpha}$ is not summable $(C, -\alpha)$ is of $(1-\alpha)$-capacity zero, if

$$\sum(a_n \cos nx+b_n \sin nx)\sim f\in L.$$

The same result is true for the series

$$\sum(a_n \sin nx-b_n \cos nx)n^{-\alpha}.$$

R. Salem (Cambridge, Mass.).

Herriot, John G. Partial-sum couplings for double Fourier series. Duke Math. J. **19**, 183–198 (1952).

Sia $f(t, u)$ una funzione periodica di periodo 2π rispetto a ciascuna delle variabili, sommabile nel quadrato $(-\pi, \pi; -\pi, \pi)$ e sia $s_{mn}(x, y)$ la somma rettangolare parziale della serie di Fourier della $f(t, u)$ nel punto (x, y). L'A. definisce il limite in senso stretto di una successione doppia nel modo seguente: si dirà che una successione doppia s_{mn} tende in senso stretto verso il limite s quando, preso comunque un numero reale $\lambda\geqq 1$, si ha $s=\lim_{m,n\to\infty} s_{mn}$ (con $1/\lambda\leqq m/n\leqq\lambda$).

L'A. considera le espressioni

$$\kappa_{mn}(x, y; p, q; h, k)$$
$$=\frac{1}{4}\left[s_{mn}(x+h, y+k)+(-1)^{p-1}s_{mn}\left(x+h+\frac{\pi p}{m}, y+k\right)\right.$$
$$+(-1)^{q-1}s_{mn}\left(x+h, y+k+\frac{\pi q}{n}\right)$$
$$\left.+(-1)^{p+q}s_{mn}\left(x+h+\frac{\pi p}{m}, y+k+\frac{q\pi}{n}\right)\right],$$

ove p, q sono numeri interi.

Se $\lim_{m\to\infty} h_m=0$, $\lim_{n\to\infty} k_n=0$ e la funzione $f(t, u)$ è continua in un punto (x, y), la successione $\kappa_{mn}(x, y; p, q; h_m, k_n)$ tende in senso stretto verso $f(x, y)$. Se $\lim_{n\to\infty} k_n=0$, $\lim_{m\to\infty} h_m=0$ ed esistono i limiti per t, $u\to 0$ delle quattro funzioni (1) $f(x\pm|t|, y\pm|u|)$, l'A. trova un'espressione del limite in senso stretto della successione $\kappa_{mn}(x, y; p, q; h_m, k_n)$ dipendente da p, q, dal $\lim_{m\to\infty} mh_m$, dal $\lim_{n\to\infty} nk_n$, dai limiti delle funzioni (1). Se esistono i limiti di certe combinazioni lineari delle funzioni (1), l'A. ricava da essi i limiti di certe combinazioni lineari delle κ_{mn}. *E. De Giorgi.*

Fast, H. On singular periodic functions. Colloquium Math. **2** (1951), 264–270 (1952). (Russian)

S. Hartman, H. Steinhaus, and H. Fast defined in a previous paper [Colloquium Math. **1**, 297–304 (1948); these Rev. **11**, 28] the notion of "singular" periodic functions: the continuous periodic function $f(x)$ with smallest positive period ω is said to be "singular" if there exists, for every $\epsilon_0>0$, an ϵ $(0<\epsilon<\epsilon_0)$ and a corresponding real number $\tau(\epsilon)$ such that $|f(x+\tau(\epsilon))-f(x)|<\epsilon$ $(-\infty<x<\infty)$ and

$$|\tau(\epsilon)-n\omega|\geqq\delta(\epsilon) \quad (n=0, \pm1, \pm2, \cdots),$$

where $\delta(\epsilon)$ denotes the least upper bound of those $\delta>0$ for which $|x_1-x_2|<\delta$ implies $|f(x_1)-f(x_2)|<\epsilon$. In the present paper it is shown that if the continuous periodic function $f(x)$ has everywhere a continuous derivative, then $f(x)$ cannot be singular. On the other hand, an example is given of a continuous periodic function $f(x)$ with (smallest positive) period ω, which is continuously derivable for $x\neq n\omega$ $(n=0, \pm1, \cdots)$, has one-sided derivatives also for $x=n\omega$, but these are unequal. *B. Sz.-Nagy* (Szeged).

Marčenko, V. A. Methods of summation of generalized Fourier series. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov Mat. Obšč. (4) **20**, 3–32 (1950). (Russian)

A set E of real numbers is called a relatively dense interval set, when there exist two positive numbers l and δ such that every interval of length l contains an interval of length δ belonging to E. The additive group of real numbers is organized as a topological group Ω with all symmetric relatively dense interval sets as neighbourhoods of zero. The class of functions continuous on Ω is identical with the class of Levitan-almost periodic (L.a.p.) functions, while the class of functions uniformly continuous on Ω is the class of ordinary almost periodic functions. By means of Banach limits the author introduces a mean value for every bounded function and extends the definition to certain unbounded functions. Every measurable function $f(x)$ which has a mean value has a Fourier series, and the author proves that this Fourier series is summable with sum $f(x)$ in every continuity point (in the sense of Ω). Fejér-Bochner summation can be applied if $f(x)$ is bounded, but de la Vallée Poussin summa-

tion can be applied, even if $f(x)$ is not bounded. Let $E(\alpha)$, $0<\alpha<\infty$, be a system of neighbourhoods of zero decreasing continuously to the empty set when $\alpha\to 0$ and satisfying the condition $E(\alpha)+E(\beta)\subset E(\alpha+\lambda(\beta))$ where $\lambda(\beta)\to 0$ when $\beta\to 0$. A point, where the limit

$$\limsup_{\alpha\to 0} M_{E(\alpha)}\{|f(x)-f(x+t)|\}$$

can be arbitrarily small by convenient choice of the system $E(\alpha)$, is called a Lebesgue point of $f(x)$. The author proves that the Fourier series determines $f(x)$ uniquely at every Lebesgue point.

In a supplement to this paper the author repeats in a slightly generalized form a direct proof by N. Bogoljubov [Krylov and Bogoljubov, New methods of nonlinear mechanics . . . (Russian), Moscow-Leningrad, 1934, p. 62] of the approximation theorem for ordinary almost periodic functions. The arithmetic properties of the translation numbers, which are essential for the discussion of L.a.p. functions, are derived from this theorem. *H. Tornehave.*

Marčenko, V. A. On functions which are normal relative to a symmetric displacement operation. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 33–42 (1950). (Russian)

Let $f(x)$ be a continuous, complex-valued function of a real variable x, $-\infty<x<\infty$. The operation

$$L_s f(x)=\tfrac{1}{2}(f(x+s)-f(x-s))$$

is called a generalized translation, and $f(x)$ is called normal with respect to generalized translations if every sequence of functions $L_{s_\nu}f(x)$, $\nu=1, 2, \cdots$, has a uniformly convergent subsequence. The author proves that the class of functions normal in this sense, is the class of functions $ax+\varphi(x)$ where a is constant while $\varphi(x)$ is an ordinary almost periodic function. *H. Tornehave* (Lyngby).

Tandori, Károly. Über die Cesàrosche Summierbarkeit der orthogonalen Reihen. Acta Sci. Math. Szeged **14**, 85–95 (1951).

Let $\int_a^b \varphi_i(x)\varphi_j(x)=0$ $(i\neq j)$, $=1$ $(i=j)$. Let

$$K_n^r(t,x)=\frac{1}{A_n^r}\sum_{k=0}^{n}A_{n-k}^r\varphi_k(t)\varphi_k(x),\quad L_n^r(x)=\int_a^b |K_n^r(t,x)|\,dt,$$

where $A_n^r=\Gamma(n+r+1)/\Gamma(r+1)\Gamma(n+1)$. The following theorem was substantially obtained (for r an integer) by S. Kaczmarz: If $r>0$ and $L_n^r(x)\leqq v_n^2$ in a set E of positive measure in (a, b), where $v_n\uparrow\infty$, and if $\sum a_k^2 v_k^2<\infty$, then $\sum a_k\varphi_k(x)$ is summable (C, α), $\alpha>0$, almost everywhere in E [Studia Math. **1**, 87–121 (1929), pp. 100–103; **5**, 24–28 (1934), pp. 26–28]. The author points out that there was a gap in Kaczmarz's argument, and gives a new proof.

L. S. Bosanquet (London).

Sunouchi, Gen-ichirô, and Yano, Shigeki. Notes on Fourier analysis. XXXIX. Convergence and summability of orthogonal series. Proc. Japan Acad. **26**, no. 7, 10–16 (1950).

Let $\int_a^b \varphi_i(x)\varphi_j(x)dx=0$ $(i\neq j)$, $=1$ $(i=j)$. Let $\lambda_n^{(\alpha)}=n^{-\alpha}$ $(0<\alpha<1)$, $=\{\log(n+1)\}^{-1}$ $(\alpha=0)$, and let $N_n^{(\alpha)}(x)$ denote the nth $(C, -\alpha)$ mean of the series $\sum\lambda_n^{(\alpha)}a_n\varphi_n(x)$. Then $\int_a^b \sup_n|N_n^{(\alpha)}(x)|^2dx\leqq A\sum a_n^2$. The result for $\alpha=0$ is due to L. Kantorovitch [C. R. (Doklady) Acad. Sci. URSS (N.S.) **14**, 537–540 (1937)]. *L. S. Bosanquet* (London).

Bertolini, Fernando. Il teorema della trasformata di Laplace d'ordine $\alpha>-1$. Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 146–155 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 316, 10 pp. (1951).

Let $F(t)$, defined for $t\geqq 0$, be summable in every finite subrange. A familiar property of the Laplace transformation is as follows: If the improper integral $\int_0^\infty \exp(-st)F(t)dt$ converges for $s=s_0$, then it also converges for any s of real part exceeding the real part of s_0. The analogue of this theorem is here established for the Laplace transform based on the following summability concept for improper integrals: If $G(t)$ is summable in every finite range and $\alpha>-1$, then

$$\int_0^\tau G(t)\left(1-\frac{t}{\tau}\right)^\alpha dt,\quad \tau>0, \tag{1}$$

is known to exist for almost all real τ. The improper integral of order α is defined as

$$(\alpha)\int_0^\infty G(t)dt=\lim_{\tau\to+\infty}\int_0^\tau G(t)\left(1-\frac{t}{\tau}\right)^\alpha dt,$$

provided the limit exists as $\tau\to+\infty$ over the set for which (1) exists. The generalized transformation

$$(\alpha)\int_0^\infty \exp(-st)F(t)dt$$

is now shown to enjoy the classical property mentioned above. *I. J. Schoenberg* (Philadelphia, Pa.).

McKean, H. P., Jr. A new proof of the completeness of the Hermite functions. Amer. Math. Monthly **59**, 621–622 (1952).

Polynomials, Polynomial Approximations

Raljević, Š. Répartition et construction des zéros d'un polynome du troisième ordre et de sa dérivée. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 167–172 (1952). (Serbo-Croatian. French summary)

The author considers the set of points in the complex plane consisting of the zeros of a cubic polynomial, the zeros of its derivative, and certain other points which enter naturally in the algebraic solution of the cubic, and he proves for this system several interesting geometric theorems. It is shown that the general solution of the cubic equation yields as a corollary the following classic theorem from geometry. On each of the sides of an arbitrary triangle, place the base of a 30° isosceles triangle with vertex directed outward (inward), then the vertices of the three isosceles triangles are the vertices of an equilateral triangle.

A. W. Goodman (Lexington, Ky.).

Gaspar Teixeira, José. Some applications of the analytic theory of polynomials. Gaz. Mat., Lisboa **12**, no. 50, 77–80 (1951). (Portuguese)

Si discute la posizione nel piano di Gauss delle radici d'un polinomio di secondo grado dipendente linearmente da due parametri, al fine di riconoscere il comportamento all'infinito degli integrali dell'equazione differenziale avente quel polinomio come caratteristico. Si riferiscono teoremi di Cauchy, Hurwitz, Schur, Kakeya e dell'autore stesso, sulla posizione delle radici di un polinomio di grado qualsiasi, e se ne generalizza uno.

Un'altra generalizzazione (teorema I) è erronea, come prova il polinomio $f(x)=ix^2+(1+i)x+5$ (avente le radici $1+2i$ e $-2-i$); il quale ne verifica le ipotesi ma non la tesi: la ragione dell'errore risiede nel fatto che, ridotta l'equazione ad avere i coefficienti reali mediante una rotazione del piano della variabile x attorno all'origine, l'applicazione del teorema di Hurwitz può dare solo le condizioni necessarie e sufficienti affinché le radici stiano da una stessa parte rispetto a quella retta che a rotazione avvenuta si sovrappone all'asse immaginario. *F. Bertolini* (Roma).

Markovitch, D. On the composite polynomials. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 11–14 (1951). (Serbo-Croatian summary)

The author states the theorem that if all the zeros of a polynomial $f(z)=\sum_0^n\binom{n}{k}a_kz^k$ lie in the circle $|z|\leqq R$, then all the zeros of the polynomial $h(z)=\sum_0^n\binom{n}{k}a_k\epsilon_kz^k$ lie in the circle $|z|\leqq MR$ where $M=\max|\epsilon_k/\epsilon_n|^{1/(n-k)}$ for $k=0, 1, \cdots, n-1$. He then applies the theorem to derive the well-known Cohn-Egervàry composition theorem. [Reviewer's note: The author's proof seems to be valid only for $R\geqq r$ where r is the positive root of the equation $\sum_0^{n-1}\binom{n}{k}|a_k|z^k-|a_n|z^n=0$.] *M. Marden* (Milwaukee, Wis.).

Atkinson, F. V. Über die Nullstellen gewisser extremaler Polynome. Arch. Math. **3**, 83–86 (1952).

Let $q_n(x)$ be the polynomial of nth degree having the highest term x^n for which

$$\int_a^b|q_n(x)|^k\rho(x)dx=\min;$$

$\rho(x)$ is a given weight function in the finite interval a, b; $k>1$. Let c be real; it is shown that the polynomial $q_n(x)+cq_m(x)$, $m\leqq n$, has at least m sign variations in a, b. *G. Szegö.*

Berghuis, J. Truncated power-series. Computation Dept., Math. Centre, Amsterdam, Rep. **R 173**, i+11 pp. (1952).

The polynomials obtained by truncating the Maclaurin series for e^x, $\cos x$, $\sin x$, and certain Bessel functions are studied. Typical results for the polynomials associated with e^x are: (a) those of even degree have no real zeros, those of odd degree exactly one; (b) any two polynomials of degrees n and $n-2$ intersect in the two points $x=0$ and $x=-n$; (c) two polynomials of even degrees $2n$ and $2m$, $n\neq m$, have exactly two real intersections, one of which is always $x=0$ and the other of which tends to negative infinity as n tends to infinity. Analogous but more complicated results are obtained for the polynomials associated with the other functions. Asymptotic formulas are also obtained for the real zeros of the polynomials. *P. W. Ketchum.*

Tomić, Boško. Quelques formules symboliques pour les polynomes de Bernoulli. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 88–91 (1952). (Serbo-Croatian. French summary)

Special Functions

Blanuša, D. Eine Verallgemeinerung des Integralkosinus. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 129–132 (1952). (Serbo-Croatian. German summary)

The generalisation in question is $\int_z^\infty t^\nu\cos t dt$ and the principal result, the expansion of this in ascending powers of x. [The author seems unaware of the connection with the well-known incomplete gamma function, or of the fact that this integral has been studied, for instance, by E. Böhmer in "Differenzengleichungen und unbestimmte Integrale" [Koehler, Leipzig, 1939], by H. Bateman [Proc. Nat. Acad. Sci. U. S. A. **32**, 70–72 (1946); these Rev. **7**, 442], and by E. Kreyszig [Acta Math. **85**, 117–181 (1951); these Rev. **12**, 825].] *A. Erdélyi* (Pasadena, Calif.).

Toscáno, Letterio. Funzioni generatrici di particolari polinomi di Laguerre e di altri da essi dipendenti. Boll. Un. Mat. Ital. (3) **7**, 160–167 (1952).

Using the customary notation $L_n^{(\alpha)}(x)$ for the generalized Laguerre polynomials, the generating series

$$\sum_{n=0}^{\infty}t^nL_n^{(\alpha-2n)}(x),\quad\sum_{n=0}^{\infty}t^nL_n^{(\alpha+n)}(x)$$

are computed in terms of elementary functions. The method is based on Lagrange's formula. Other related expansions are discussed; for instance,

$$\sum t^n[\Gamma(n+1+\tfrac{1}{2}\alpha)]^{-1}L_n^{(\alpha+n)}(x)$$

is expressed in terms of a Bessel function, and

$$\sum\alpha(\alpha+2n)^{-1}t^nL_n^{(\alpha+n)}(x)$$

in terms of ${}_1F_1$. Finally, the relation of generating functions of polynomials $\{A_n(x)\}$, $\{B_n(x)\}$ is discussed for which the identity $e^xA_n(x)=\{e^xB_n(x)\}^{(n)}$ holds. *G. Szegö.*

Toscano, Letterio. Su una disuguaglianza relativa ai polinomi di Hermite. Boll. Un. Mat. Ital. (3) **7**, 171–173 (1952).

We use the notation $e^{-x^2/2}H_n(x)=(e^{-x^2/2})^{(n)}$. The author proves that the expression $H_n^2(x)-H_{n-1}(x)H_{n+1}(x)$ attains its minimum for $x=0$. *G. Szegö* (Stanford, Calif.).

*Tricomi, Francesco G. **Lezioni sulle funzioni ipergeometriche confluenti.** Gheroni, Torino, 1952. 284 pp. 2000 Lire.

Confluent hypergeometric series were introduced by Kummer more than a century ago. In 1904, E. T. Whittaker proposed new definitions and notations which clearly exhibit the symmetry and the transformation properties of confluent hypergeometric functions, and he also showed that a large number of special functions (Bessel functions, Laguerre polynomials, error functions, and many others) are particular confluent hypergeometric functions. Since then a sizable literature has grown up around these functions and more and more applications were found. Several monographs were planned but, as far as the reviewer knows, none appeared in print. Thus the book under review is the first one in the field; and it derives added significance from its author's gift for exposition, and from his many contributions in recent years to the theory of confluent hypergeometric functions.

The book is based on a course of lectures given at the University of Torino, and this origin is noticeable throughout the book. The presentation is well planned, leisurely, detailed (very few computations are "left to the reader"), and self-contained, with several digressions on topics with which the student may not be familiar. There is no attempt at all-inclusiveness but the topics which have been included are discussed rather fully. Books and memoirs are quoted when they are needed in the text; otherwise there is no bibliography, and there is no index. All in all, we have a well written and eminently readable introduction to confluent

hypergeometric functions before us which is not, and does not attempt to be, a work of reference. Prerequisites for reading this book are advanced calculus and the more elementary parts of complex function theory.

The confluent hypergeometric equation is the differential equation

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=0 \tag{1}$$

in which a and c are constant parameters. That solution of (1) which is regular, and equal to unity, at the origin is here denoted by $\Phi(a, c; x)$ (Kummer's series). A second solution, which vanishes at $+\infty$ and corresponds to Whittaker's function $W_{k,m}$, is denoted by $\Psi(a, c; x)$.

Chapter I contains preliminaries. Since the Laplace transformation is used extensively in investigating confluent hypergeometric functions, the author gives, in about 15 pp., a brief summary of the relevant theorems. Then follow some 10 pp. on the analytic theory of linear differential equations, and about 25 pp. on the gamma function.

Chapter II. It is first shown that any linear differential equation of the second order whose coefficients are linear functions of x, may be reduced to (1), and then some other differential equations reducible to (1) are given. If c is not an integer, the general solution of (1) is a linear combination of $\Phi(a, c; x)$ and $x^{1-c}\Phi(a-c+1, 2-c; x)$. The properties of Φ, or rather of

$$\Phi^*(a, c; x)=\frac{1}{\Gamma(c)}\Phi(a, c; x)$$

which is an entire function of all three of its arguments, are discussed, the discussion including Kummer's transformation, recurrence and differentiation formulas, Laplace transforms, and more. In case $c=2a$, there is a connection with Bessel functions, and in case a is a negative integer, with Laguerre and Hermite polynomials. This chapter concludes with the author's well-known expansion of Φ in series of Bessel functions.

Chapter III introduces the confluent hypergeometric function of the second kind Ψ, integral representations for both Φ and Ψ, gives the relations connecting these two functions, recurrence relations, etc., for Ψ, and brings a discussion of the asymptotic behaviour of confluent hypergeometric functions. For large x, the asymptotic expansions follow from the integral representations; for large parameters and argument, the author uses his expansion in series of Bessel functions. The results are specialized for $c=2a$ when they give the usual asymptotic expansions of Bessel functions. For real a and c, the real zeros of Φ and the positive zeros of Ψ are counted, and the results are used to describe $\Phi(a, c; x)$ for real a, c, x in some detail.

Chapter IV is devoted to the incomplete gamma functions and related functions. The author puts

$$\int_0^x e^{-t}t^{\alpha-1}dt=\gamma(\alpha, x)=x^\alpha\Gamma(\alpha)\gamma^*(\alpha, x)=\Gamma(\alpha)-\Gamma(\alpha, x)$$

and points out that γ^* is the basic function, being an entire function of both α and x. The connection with confluent hypergeometric functions with $a=1$ or $c=a+1$ leads readily to most of the properties of incomplete gamma functions, except that the asymptotic behaviour of these functions in case both x and α are large merits a separate detailed investigation. There is again an investigation of real zeros and a description of the incomplete gamma functions when both x and α are real, and there are separate sections on some of the more special functions connected with the incomplete gamma function, such as the error functions, exponential integral, sine and cosine integrals, and Fresnel's integrals. The last section in this chapter gives a tabular index to most of the important particular cases of confluent hypergeometric functions.

Chapter V contains applications. The two body-problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, and water waves are discussed and impress the reader with the manifold applications of the functions treated in this book.

The book is reproduced from a typescript; according to information received from the author, a printed edition is in preparation. *A. Erdélyi* (Pasadena, Calif.).

Tricomi, Francesco G. Sulle derivate delle funzioni ipergeometriche confluenti rispetto ai parametri. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 227–233 (1952).

Derivatives of confluent hypergeometric functions with respect to the parameters are not themselves hypergeometric functions, but in exceptional cases it happens that certain combinations are again hypergeometric series. The notations $\Phi(a, c; x)$ and $\Psi(a, c; x)$ are as in the author's book on confluent hypergeometric functions reviewed above, $\Phi=\Phi_a+2\Phi_c$, etc. (subscripts indicate partial differentiation), and in the simplest case the relations discovered by the author are

$$\dot{\Phi}(a, 1; x)=-[2C+\psi(a)+\log x]\Phi(a, 1; x)-\Gamma(a)\Psi(a, 1; x)$$
$$\dot{\Psi}(a, 1; x)=-\Psi(a, 1; x)\log x$$

(C is Euler's constant). The author points out the connection between his formulas and the expressions for derivatives of Bessel functions with respect to the order, and gives also applications to incomplete gamma functions and related functions. *A. Erdélyi* (Pasadena, Calif.).

Erdélyi, A. The general form of hypergeometric series of two variables. Proc. Amer. Math. Soc. **2**, 374–379 (1951).

A formal hypergeometric series of two variables is of the form: $\sum a_{m,n}x^my^n$ for whose coefficients the ratios:

$$a_{m+1,n}/a_{m,n}=f(m, n), \quad a_{m,n+1}/a_{m,n}=g(m, n)$$

are fixed rational functions of m and n satisfying the consistency condition:

$$f(m, n+1)g(m, n)=f(m, n)g(m+1, n). \tag{1}$$

P. Appell, J. Horn, H. Mellin, L. Koschmieder, and the reviewer have considered the case where

$$a_{m,n}=\prod_i\Gamma(\alpha_i+u_im+v_in)/\Gamma(\alpha_i), \tag{2}$$

α_i being arbitrary complex constants, u_i and v_i arbitrary integers which may be positive, zero, or negative; this seems to be justified by a statement of Birkeland [C. R. Acad. Sci. Paris **185**, 923–925 (1927)] that every rational solution of (1) can be decomposed into linear factors. But Ore [J. Math. Pures Appl. (9) **9**, 311–326 (1930)] remarked that there are obvious exceptions to this rule and that thus the statement of Birkeland was not true.

The main object of this paper is to find the general form of rational functions satisfying (1); the result is that the most general form of the coefficients of a hypergeometric series is:

$$a_{m,n}=h^mk^nR(m, n)\prod_i\Gamma(\alpha_i+u_im+v_in)/\Gamma(\alpha_i)$$

where h, k are constant, $R(m, n)$ a rational function of m and n, the significance of $\prod$ being the same as in (2). For the reduction of these series to standard types, see A. Erdélyi [Proc. Roy. Soc. Edinburgh. Sect. A. 62, 378–385 (1948); these Rev. 10, 115]. *J. Kampé de Fériet* (Lille).

Sears, D. B. Two identities of Bailey. J. London Math. Soc. 27, 510–511 (1952).

These identities appeared in Proc. London Math. Soc. (3) 1, 217–221 (1951) [these Rev. 13, 327]. Sears shows that both are special cases of an earlier result of his involving four parameters [ibid. (2) 53, 158–180 (1951); these Rev. 13, 33]. *N. J. Fine* (Philadelphia, Pa.).

Slater, L. J. General transformations of bilateral series. Quart. J. Math., Oxford Ser. (2) 3, 73–80 (1952).

Using results of D. B. Sears on transformation of unilateral basic hypergeometric series [Proc. London Math. Soc. (2) 53, 158–180, 181–191 (1951); these Rev. 13, 33, 129] the author derives a formula which expresses the general bilateral series ${}_M\Psi_M$ in terms of M other series of the same type. There are similar results for well-poised ${}_M\Psi_M$. The corresponding formulas for ordinary hypergeometric series are obtained by letting $q\to 1$; the main result is extended by means of an integral of Barnes type. *N. J. Fine.*

Bragard, L. Sur quelques formules relatives aux harmoniques sphériques. Bull. Soc. Roy. Sci. Liège 21, 46–70 (1952).

Bragard, L. Sur quelques intégrales doubles relatives aux harmoniques sphériques. I. Bull. Soc. Roy. Sci. Liège 21, 158–178 (1952).

In both papers the author discusses (and evaluates) various integrals of the form

$$\int\int Y_n(\theta, \varphi) S_m(\theta, \varphi) d\sigma$$

where Y_n and S_m are surface harmonics and $d\sigma$ is the surface element of the unit sphere. In most of the formulas Y_n is arbitrary and S_m is of special nature. *G. Szegö.*

Reissner, Eric. Reihenentwicklung eines Integrals aus der Theorie der elastischen Schwingungen. Math. Nachr. 8, 149–153 (1952). Read the corresponding note p. 278.

Let

$$f_1(a_0) = \mathrm{Re}\left\{\int_0^\infty J_1(a_0\theta) F(\theta^2)\theta^{-1} d\theta\right\}$$

where $a_0>0$, J_1 is the Bessel function of the first kind of order unity, $0<\delta^2<1$, and

$$F(\theta^2) = \frac{\theta(\theta^2-\delta^2)^{1/2}}{(2\theta^2-1)^2 - 4\theta^2(\theta^2-1)^{1/2}(\theta^2-\delta^2)^{1/2}}.$$

Integration may be restricted to the range $\theta>\delta$, and in this range $F(\theta^2)$ may be expanded in descending powers of θ^2. This leads to an expansion of f_1 in ascending powers of a_0. *A. Erdélyi* (Pasadena, Calif.).

Lambe, C. G. Lamé-Wangerin functions. Quart. J. Math., Oxford Ser. (2) 3, 107–114 (1952).

The functions discussed in this paper are those solutions of Lamé's differential equations

$$\frac{d^2\lambda}{du^2} + \{h - n(n+1)\wp(u)\}\lambda = 0,$$

with $2n$ an odd integer, which are algebraic functions of $\wp(u)$. An alternative form (due to Halphén) is obtained by putting $u=2v$ and $\lambda=[\wp'(v)]^{-n}\mu(v)$. Each of these two equations can be transformed to an "algebraic form" (i.e., to a differential equation with rational coefficients which is a particular case of Heun's equation). The present paper uses both algebraic forms, derives the expressions in finite form for the solutions and also for products of solutions, and integral equations. *A. Erdélyi* (Pasadena, Calif.).

González, Mario O. On a function defined by a differential equation. Actas Acad. Ci. Lima 15, 93–97 (1952). (Spanish)

The function $\omega=\mathrm{Tan}\, z$ is defined by the differential equation $(d\omega/dz)^2=1+2\lambda\omega^2+\omega^4$ with the initial conditions $\omega(0)=0$, $\omega'(0)=1$. The author gives a power series expansion for $\omega(z)$ and for the inverse function. *A. Erdélyi.*

Sips, Robert. Les constantes caractéristiques de l'équation intégrale des fonctions de Mathieu. Bull. Soc. Roy. Sci. Liège 21, 141–157 (1952).

The author starts from the linear homogeneous integral equations for the Mathieu functions ce_n and se_n. These equations each contain, besides the well-known parameter, which occurs in the kernel, a characteristic parameter as multiplier of the integral expression. These parameters have until now hardly been the object of any worth-while study. The author shows that their values may easily be evaluated from the coefficients of the Fourier series expansions of the four classes of Mathieu functions of the first kind. Inserting known numerical values of these coefficients he obtains a short table of values for the characteristic parameters as dependent on the parameter occurring in the kernel of the integral equations. This table is illustrated by a set of curves. He then proceeds to study power series developments of the characteristic parameters as dependent on the kernel parameter. Herein he makes use of the original series developments of E. Mathieu. Upper bounds of convergence of the power series are given. He then studies the asymptotic values of the characteristic parameters at very large values of the kernel parameters, making use of known asymptotic expressions of the Mathieu functions in question and arrives at asymptotic expressions, which are tested numerically by comparison with values derived previously. Finally, relations between the different characteristic constants are given, their inverse squares are summed, and they are compared with characteristic values of the Mathieu differential equations. *M. J. O. Strutt* (Zurich).

Harmonic Functions, Potential Theory

Green, John W. Approximately subharmonic functions. Proc. Amer. Math. Soc. 3, 829–833 (1952).

D'après Sjöberg et Brelot, il existe pour une fonction semi-continue supérieurement, $<+\infty$, et majorant une fonction sousharmonique, dans un ouvert, une plus grande minorante sousharmonique [voir Brelot, J. Math. Pures Appl. (9) 24, 1–32 (1945); ces Rev. 7, 521]. Cela s'appuyait sur la propriété délicate (Cartan) de quasi-sousharmonicité de l'enveloppe supérieure d'une famille de fonctions sousharmoniques (uniformément bornées supérieurement localement). L'auteur ici arrive au résultat avec la seule notion de presque sousharmonicité de Szpilrajn. Puis il s'en sert

pour étudier la notion de fonction f ϵ-sousharmonique dans un ouvert euclidien, c'est-à-dire, semi-continue supérieurement (et ce qui est sousentendu $<+\infty$) et majorée à ϵ près dans tout domaine partiel borné ω par toute fonction finie continue dans $\bar{\omega}$, harmonique dans ω et majorant f sur la frontière ω^*. L'auteur montre l'existence d'une approximation à ϵ près par une fonction sousharmonique minorante et se demande si on pourrait faire en sorte que la différence soit semi-continue supérieurement. *M. Brelot* (Grenoble).

Beckenbach, E. F. On characteristic properties of harmonic functions. Proc. Amer. Math. Soc. **3**, 765–769 (1952).

S'appuyant sur un résultat de Saks [Bull. Amer. Math. Soc. **38**, 380–382 (1932)], l'auteur établit un critère d'harmonicité pour une fonction $u \in C'(D)$ (i.e., continuement différentiable dans un domaine plan D) ne s'annulant pas dans D: en tout point (x_0, y_0) de D on a

$$\int\int (\text{grad } u)^2 dxdy - \int u \frac{du}{dn} ds = o(r^2)$$

où l'intégrale double est étendue au cercle $C(x_0, y_0; r)$, l'intégrale simple à la circonférence de ce cercle. Pour éliminer l'hypothèse $u \neq 0$ dans D, on est conduit au problème: toute fonction $u \in C'(D)$, harmonique en tout point de D où elle n'est pas nulle, est-elle harmonique dans D ? L'auteur en donne une réponse affirmative en supposant que $\partial^2 u/\partial x^2$ et $\partial^2 u/\partial y^2$ existent et sont sommables dans D. *J. Deny* (Strasbourg).

Kuramochi, Zenjiro. Potential theory and its applications. II. Osaka Math. J. **4**, 87–99 (1952).

This is a continuation of the author's survey on a number of problems in the theory of Riemann surfaces [same J. **3**, 123–174 (1951); these Rev. **13**, 650]. The Dirichlet problem and its connection with harmonic measure, Green's function, the transfinite kernel, barriers, and accessibility of boundary points are discussed. *L. Sario* (Cambridge, Mass.).

***Lavrent'ev, M. A. The Dirichlet problem for a narrow stratum.** Trudy Mat. Inst. Steklov., v. 38, pp. 146–151. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Let D be a domain in 3-space homeomorphic to the region bounded by two concentric spheres. Let D be bounded by two continuously curved surfaces Γ and Γ_0 on which two functions, f and f_0, of class C^2 are defined, and let W be the harmonic function in D determined by the boundary conditions: $W=f$ on Γ, $W=f_0$ on Γ_0. It is assumed that through every point of Γ and Γ_0 there pass two spheres of radius r, one contained in D and the other in the complement of D, that every point A of Γ can be joined to a point A_0 of Γ_0 by a segment of the normal to Γ whose length n satisfies the inequality $0<\delta_0 \leqq n \leqq \delta$, and that a similar condition is satisfied at every point of Γ_0. Let W_n denote the normal derivative of W at a point A of Γ.

The author states an estimate for $|W_n|$ depending on r, δ_0, δ and a bound on the boundary functions f, f_0 and their derivatives up to the second order (in an appropriate coordinate system) and an estimate of the change of W_n resulting from a slight change in Γ and f in the neighborhood of a point $A' \neq A$. Assuming that f, f_0 are of class C''', and that Γ is close to Γ_0 and f to f_0, "of order 2", the author derives an approximate formula for W_n depending on the values $f(A)$, $f_0(A_0)$, on the distance $\overline{AA_0}$, on the second derivatives of f at A, and on the mean curvature of Γ at A. The error term in this formula is $O(h^3)$, h being a parameter characterizing the narrowness of the region D.

The presentation is very condensed, and the reviewer was unable to follow the details. *L. Bers.*

Garnir, H. G. Sur la formulation des problèmes aux limites dans la théorie des distributions. Bull. Soc. Roy. Sci. Liège **20**, 497–513, 639–649 (1951).

Considérons le problème de Dirichlet pour le laplacien Δ dans R^n, relativement à un domaine D de frontière $\dot{D}$ assez régulière: second membre $m(x)$, valeur au contour $u_0(x)$. Une solution $u(x)$ ayant au contour une dérivée normale intérieure $u_1=du/dn$, vérifie (formule de Green), pour toute fonction $\varphi(x) \in \mathfrak{D}$ (langage des distributions [Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; ces Rev. **12**, 31, 833]):

$$u(\Delta\varphi) = m(\varphi) + \int_{\dot{D}} \varphi u_1 dS - \int_{\dot{D}} u_0 \frac{d\varphi}{dn} dS.$$

Comme u_1 n'est pas connue, l'auteur introduit le sous espace $\mathfrak{A}$ de $\mathfrak{D}$ formé des fonctions φ vérifiant $\varphi=0$ sur $\dot{D}$; si $\mathfrak{A}^0$ est son orthogonal dans $\mathfrak{D}'$, la distribution u vérifiera l'equation avec second membre

$$\Delta u = m + \frac{d}{dn}(u_0 \delta_{(\dot{D})}) \bmod \mathfrak{A}^0$$

($\delta_{(\dot{D})}$ désignant la mesure de densité superficielle 1 portée par $\dot{D}$); et réciproquement, moyennant des conditions de régularité convenables.

L'auteur établit ce même formalisme pour des opérateurs différentiels généraux, elliptiques ou hyperboliques, pour des problèmes aux limites classiques, à partir de formules de réciprocité de Green. *L. Schwartz* (Nancy).

Garnir, H. G. Sur deux équations de la théorie des distributions. Bull. Soc. Roy. Sci. Liège **20**, 650–666 (1951).

Suite de l'article reviewé ci-dessus. L'auteur applique le formalisme qu'il a défini antérieurement au cas particulier où les données au contour sont nulles et où $m_x=\delta_x(\xi)$, distribution en x définie par la masse unité au point ξ. La solution, si elle existe, avec les conditions de support voulues, est la distribution de Green $G_x(\xi)$ relative au point ξ; si elle dépend assez régulièrement de ξ, alors, compte tenu de $S_x=\int \delta_x(\xi) S_\xi d\xi$, la solution du problème aux limites général avec second membre modifié S_x (somme du second membre m_x et d'une distribution tenant compte des données au contour) sera $\int G_x(\xi) S_\xi d\xi$: les distributions de Green relatives aux différents points ξ donnent la solution du problème général. Les conditions relatives à la régularité et au support sont examinées. Pour trouver des solutions de Green, l'auteur développe, dans le cas hyperbolique, la méthode de transformation de Laplace partielle par rapport au temps [Garnir, C. R. Acad. Sci. Paris **234**, 583–585 (1952); ces Rev. **13**, 751] et donne comme exemple l'équation des ondes. *L. Schwartz* (Nancy).

Savin, S. A. Formation of integrals of the three-dimensional Laplace equation by means of a function of four-termed arguments and some of their applications. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 621–624 (1951). (Russian)

Let α_j^i be constants ($i=1, 2$; $j=1, 2, 3$) and set

$$(\rho^i)^2 = \sum (\alpha_j^i)^2, \quad r^2 = \sum (x_j)^2.$$

Assume that $\sum \alpha_j^1 \alpha_j^2 = \rho^1 \rho^2$, and set $k^i = \sum \alpha_j^i x_j + \rho^i r^i$. (By

choosing the α's complex one can achieve that the expressions k^1/k^2 and $\rho^2k^1-\rho^1k^2$ be variable.) Let F and G be arbitrary analytic functions of one variable. Then $F(k^1/k^2)$ and $G(\rho^2k^1-\rho^1k^2)$ are harmonic in (x_1, x_2, x_3). So is $\log k$. Some examples are given. *L. Bers* (New York, N. Y.).

Maruhn, Karl. Über die Potentiale von Belegungen auf unendlichen Flächen. Math. Nachr. **8**, 239–248 (1952).

The paper is devoted to obtaining bounds for the Newtonian potentials of single and double layers, and their derivatives, when the layers lie on a surface which extends to infinity. The surface is assumed to be expressible in the form $z=f(x, y)$, when f has bounded second derivatives. In the case of a single layer, the density is assumed to be $O(1+R)^{-\lambda}$ where R is the distance from the origin and $\lambda>1$. Also the density satisfies a Lipschitz condition which becomes more severe as the pair of points recedes to infinity; namely the multiplicative constant is $O(1+\bar{R})^{-\lambda}$ where $\bar{R}$ is the smaller of the two distances from the origin. A number of estimates of the potential U and any first derivative DU are obtained, depending on λ, of which the following are examples: $U=O(R^{1-\lambda})$ $(1<\lambda<2)$, $U=O(R^{-1}\log R)$ $(\lambda=2)$, $U=(\text{total mass})/R+O(R^{1-\lambda})$ $(2<\lambda<3)$, $DU=O(R^{-\lambda}\log R)$ $(1<\lambda\leqq 2)$, etc. In the case of a double layer, the first derivatives of the density are assumed to be bounded and to satisfy Lipschitz conditions of the sort above, with $-1-\lambda$ instead of λ. For the potential V estimates are obtained including $V=O(R^{-\lambda}\log R)$ $(0<\lambda\leqq 2)$, $V=O(R^{-\lambda})$ $(\lambda>2)$, $DV=O(R^{-1-\lambda})$ $(0<\lambda<2)$, etc. *J. W. Green.*

Synge, J. L. Conditions satisfied by the expansion and vorticity of a viscous fluid in a fixed container. Quart. Appl. Math. **9**, 319–322 (1951).

The author considers the following problem: given a scalar θ and a vector $\mathbf{w}$ defined and sufficiently smooth throughout a simply-connected region V, what is a necessary and sufficient condition that θ be the divergence and $\mathbf{w}$ the curl of a field $\mathbf{v}$ which vanishes upon the boundary? While the problem is one of pure potential theory, its application to the flow of a material which adheres to bounding surfaces is immediate: θ is the expansion and $\mathbf{w}$ the vorticity. Guided by a result of Hamel [Göttingen Nachr. Math.-Phys. Kl. 1911, 261–270], the author had previously solved the corresponding problem in the plane [same Quart. **8**, 107–108 (1950); these Rev. **12**, 59]. He now obtains the condition

$$\int_V (\theta\phi+\mathbf{w}\cdot\mathbf{b})dV=0,$$

where $\mathbf{b}$ and ϕ are related through grad $\phi=$curl $\mathbf{b}$, but otherwise are arbitrary. The author remarks that his proof of sufficiency is "not mathematically rigorous," since it rests upon a lemma which he is content to make "plausible." In a footnote added in proof he remarks that his result was given in prior papers by the reviewer [see the following review] and van den Dungen [same Quart. **9**, 203–204 (1951); these Rev. **13**, 459], but that the latter's proof of sufficiency is faulty. No conditions of regularity on the domain or the variables are formulated by the author. [These, along with a sufficiency proof which does not rest on plausibility arguments, will be found in the second cited note of the reviewer, in the first of which is an alternative solution to the same problem.]

C. Truesdell (Bloomington, Ind.).

Truesdell, Clifford. Caractérisation des champs vectoriels qui s'annulent sur une frontière fermée. C. R. Acad. Sci. Paris **232**, 1277–1279 (1951).

Truesdell, Clifford. Analogue tri-dimensionnel au théorème de M. Synge concernant les champs vectoriels plans qui s'annulent sur une frontière fermée. C. R. Acad. Sci. Paris **232**, 1396–1397 (1951).

The author solves the following problem whose implications for the theory of a viscous fluid filling a bounded volume are obvious: given a vector field $\mathbf{w}$ and a scalar field θ in a 3-dimensional bounded domain V, find necessary and sufficient conditions for the existence of a vector field $\mathbf{v}$ vanishing on the boundary S of V and such that curl $\mathbf{v}=\mathbf{w}$ and div $\mathbf{v}=\theta$.

The solution for a 2-dimensional domain and the particular case $\theta=0$ goes as far back as Hamel [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1911, 261–270]: there is some lack of rigor in the demonstration [see A. Weinstein, Amer. J. Math. **63**, 615–618 (1941); these Rev. **3**, 244] and the same problem has been solved again by the reviewer [Math. Mag. **21**, 74–79 (1947); Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 11–18 (1948); these Rev. **9**, 433, 509]. The solution has been given for an arbitrary θ by J. L. Synge [Quart. Appl. Math. **8**, 107–108 (1950); these Rev. **12**, 59].

The 3-dimensional case with $\theta=0$ has been solved by R. Berker [C. R. Acad. Sci. Paris **228**, 1630–1632 (1949); Bull. Sci. Math. (2) **73**, 163–176 (1949); these Rev. **11**, 107, 357].

The main result of the present papers is: let us suppose that a) θ satisfies a Hölder condition in V; b) $\mathbf{w}$ has derivatives of the first order in V satisfying Hölder conditions in V; c) div $\mathbf{w}=0$; there exists a vector field $\mathbf{v}$, vanishing on S, having continuous derivatives of the first order in V and such that div $\mathbf{v}=\theta$, curl $\mathbf{v}=\mathbf{w}$ if and only if

$$\int_V [\psi\theta+\mathbf{f}\times\mathbf{w}]dV=0$$

for every vector field $\mathbf{f}$ such that: curl $\mathbf{f}=$grad ψ, $\nabla^2\psi=0$.

J. Kampé de Fériet (Lille).

Truesdell, C. Vorticity averages. Canadian J. Math. **3**, 69–86 (1951).

In parts I and II, the author surveys all the known vorticity integrals for the motion of a fluid which adheres to the bounding surfaces. Part I is devoted to the 2-dimensional motion of an inviscid fluid: integrals of Poincaré and Hamel [see the preceding review] and part II to the 3-dimensional case: a) inviscid case: integrals of Lamb, Poincaré, R. Berker, Moreau; b) general case: A. Föppl, Jaffé, and R. Berker [C. R. Acad. Sci. Paris **228**, 1630–1632 (1949); Bull. Sci. Math. (2) **73**, 163–176 (1949); these Rev. **11**, 107, 357]. In part III, the author gives conservation theorems for the vorticity of a very general type. The fluid is not supposed inviscid; the only hypothesis is that the motion is continuously differentiable; the analysis is purely kinematical and thus the results stated are independent of whatever dynamical characteristics the medium may possess. Using the polyadic notation of Gibbs and Wilson [Vector analysis, 5th ed., Yale Univ. Press, 1925] he puts: $\{\mathbf{b}^n\mathbf{c}\}$ for the completely symmetrical polyadic:

$$\{\mathbf{b}^n\mathbf{c}\}=\mathbf{b}^n\mathbf{c}+\mathbf{b}^{n-1}\mathbf{c}\mathbf{b}+\mathbf{b}^{n-2}\mathbf{c}\mathbf{b}^2+\cdots+\mathbf{c}\mathbf{b}^n$$

and defines the nth symmetrical moment W_n of the vorticity

$$W_n=\int_V \{\mathbf{r}^n\mathbf{w}\}dV.$$

The predominant effect of the boundaries upon vorticity are illustrated by the following proposition: all the moments are independent of the motion at interior points, being completely determined by the normal component w_n of the vorticity upon the boundaries.

The main results are two general conservation theorems, whose first reads as follows: if upon any finite boundary of a continuously differentiable motion the normal component w_n of the vorticity be zero, while in any portion extending to infinity the radial w_r component satisfy the condition $r^{k+2}w_r \to 0$, then the first $k+1$ symmetrical moments of vorticity vanish. *J. Kampé de Fériet* (Lille).

Maple, Clair G. Bounds for the derivatives of the solution of the Neumann problem. Proc. Iowa Acad. Sci. **57**, 307–313 (1950).

L'autore si vale del metodo dell'iperciclo nello spazio funzionale, di Prager e Synge [Quart. Appl. Math. **5**, 241–269 (1947); questi Rev. **10**, 81], per determinare dei limiti per le derivate della soluzione del problema di Neumann. *L. Amerio* (Milano).

Pachale, Helmut. Über ein ebenes nichtlineares biharmonisches Randwertproblem. Math. Nachr. **7**, 187–212 (1952).

Let R be a simply connected, bounded domain in the plane, S its boundary, R_a the complement of $R+S$. It is assumed that the boundary S has a tangent vector at each of its points which is continuous and satisfies a uniform Hölder condition on S for some λ, $0<\lambda<1$. Let $F(p, \mu, \nu; C)$, $G(p, \mu, \nu; C)$ be two functions defined for p on S, μ and ν functions on S for which $|\mu(p)|$, $|\nu(p)| \leqq \Omega^*$, and the positive number C, $0 \leqq C \leqq C^*$. If $\Omega = \sup_{p \in S} \{\max(|\mu(p)|, |\nu(p)|)\}$, we suppose that

$$|F(p, \mu, \nu; C)|, \quad |G(p, \mu, \nu; C)| \leqq \hat{M}(C) + M(C, \Omega)\Omega$$

where $\hat{M}(C)$, $M(C, \Omega)$ are continuous, non-decreasing functions of C and Ω separately for $0 \leqq C \leqq C^*$, $0 \leqq \Omega \leqq \Omega^*$, $M(C) \to 0$ as $C \to 0$, $M(C, \Omega) \to 0$ as $C^2 + \Omega^2$ goes to 0. Further, it is supposed that $|F(p, \bar{\mu}, \bar{\nu}; C) - F(p, \mu; \nu; C)| \leqq \bar{M}(C, \Omega)\hat{\Omega}$, where $\hat{\Omega} = \sup_{p \in S} \{\max(|\bar{\mu}(p) - \mu(p)|, |\bar{\nu}(p) - \nu(p)|)\}$, $\hat{\Omega} \leqq 2\Omega$, and $\bar{M}(C, \Omega)$ behaves like $M(C, \Omega)$. A similar condition holds for G. This paper is concerned with the existence of solutions of the biharmonic equation, $\Delta^2 U = 0$ satisfying non-linear boundary conditions which are perturbations of the homogeneous conditions $U_s = 0$, $U_y = 0$ on S, namely

$$U_s = F(p, U_s, U_y; C), \quad U_y = F(p, U_s, U_y; C).$$

The solutions are sought in R and in R_a with the additional condition that U vanish at infinity like $1/r^2$. It is shown under the assumption of certain integral integrability conditions that for sufficiently small values of C the solutions exist and are unique up to a constant. More general boundary conditions are considered. The proofs are carried through by successive approximations. *F. Browder.*

Pachale, Helmut. Über ein räumliches nichtlineares biharmonisches Randwertproblem. Math. Nachr. **8**, 79–91 (1952).

A generalization is given to domains in three-space of the existence and uniqueness theorems for biharmonic functions satisfying non-linear boundary conditions established by the author for domains in the plane in the paper reviewed above. *F. Browder* (Boston, Mass.).

Differential Equations

Katō, Tizuko. Sur les points singuliers des équations différentielles ordinaires du premier ordre. Nat. Sci. Rep. Ochanomizu Univ. **2**, 13–17 (1951).

Let $P(x, y)$ and $Q(x, y)$ be coprime polynomials in y with coefficients regular for $|x| < \Delta$, set $P_0(y) = P(0, y)$, $Q_0(y) = Q(0, y)$, and let $P_0^c(y)/Q_0^c(y) = P_0(y)/Q_0(y)$ with $P_0^c(y)$ and $Q_0^c(y)$ coprime. Let $y(x)$ satisfy the differential equation $x^{\sigma+1}y' = P(x, y)/Q(x, y)$ with σ a positive integer. The object of this paper is to prove that, if no combination of residues of $Q_0(y)/P_0(y)$ has a purely imaginary sum, then, as $x \to 0+$, either $y(x)$ approaches a zero of $P_0(y)$, or there is a decreasing sequence x_n such that $y(x_n)$ approaches a multiple zero of $P_0^c(y)$. *F. A. Ficken.*

Katō, Tizuko. On singular points of ordinary differential equations of the first order. Nat. Sci. Rep. Ochanomizu Univ. **1**, 17–21 (1951).

Hukuhara has shown [Mem. Fac. Engineering, Kyusyu Imp. Univ. **8**, 203–247 (1937), p. 233] that the solution of $xy' = yF(x, y)$, under suitable assumptions and in a suitable region, can be expressed as $y = \sum \psi_k(u)x^k$ where u satisfies the equation $xu' = u^{n+1}(a + bu^n)$ with a and b constants, $a \neq 0$, and n a positive integer. The object here is to prove this theorem with the aid of a recent theorem of Hukuhara's on fixed points [cf., presumably, Jap. J. Math. **20**, 1–4 (1950); these Rev. **13**, 561]. *F. A. Ficken.*

Stebakov, S. A. Qualitative investigation of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ by means of isoclines. Doklady Akad. Nauk SSSR (N.S.) **82**, 677–680 (1952). (Russian)

The autonomous system $x' = P(x, y)$, $y' = Q(x, y)$ ($' = d/dt$) is considered in the neighborhood of an isolated singular point. By constructing certain polygonal paths and considering the direction field defined by the above system it is shown that some qualitative results can be stated concerning the solution paths of the system near the singular point. The constructions do not seem to differ much from the standard ones used to prove the Poincaré-Bendixson theorem. *E. A. Coddington* (Los Angeles, Calif.).

Grobman, D. M. Systems of differential equations analogous to linear ones. Doklady Akad. Nauk SSSR (N.S.) **86**, 19–22 (1952). (Russian)

The author considers the vector-matrix system of differential equations, $dx/dt = Ax + f(t, x)$, where $f(t, 0) = 0$, $\|f(t, x) - f(t, z)\| \leqq g(t)\|x - z\|$, $\|x\| = \sum_{i=1}^{n} |x_i|$, and establishes under various assumptions concerning $g(t)$, connections between the asymptotic behavior of x and y as $t \to \infty$, where $dy/dt = Ay$. An investigation along similar lines is contained in a paper by the reviewer [Trans. Amer. Math. Soc. **62**, 357–386 (1947); these Rev. **9**, 436]. *R. Bellman* (Santa Monica, Calif.).

Richard, Ubaldo. Sulla rappresentazione asintotica degli estremi delle soluzioni di equazioni differenziali lineari del 2° ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 382–387 (1952).

Let $y(x)$ be a solution of the differential equation $d[p(x)dy/dx]/dx + q(x)y(x) = 0$. It is assumed that $p(x) > 0$, $q(x) > 0$, for $x \geqq a$, that $y(x)$ takes on extreme values for $x = x_n$ ($n = 1, 2, \cdots$), and that $\lim_{n \to \infty} x_n = \infty$. The zeros of $y(x)$ may be denoted by ξ_n. If the function

$$w(x) = \tfrac{1}{2}p(x)d[\{p(x)q(x)\}^{-1/2}]/dx$$

tends monotonically to zero, as $x\to\infty$, the author shows that the sequences

$$\rho_1(x_n)=\{p(x_n)q(x_n)\}^{1/2}\{1+w(x_n)\}^{-1}y^2(x_n),$$
$$\rho_2(x_n)=\{p(x_n)q(x_n)\}^{1/2}\{1-w(x_n)\}^{-1}y^2(x_n)$$

tend monotonically, from opposite sides, to the same positive constant. An analogous statement holds for the sequence $y'(\xi_n)$. The information on the asymptotic behavior of $y(x_n)$ and $y'(\xi_n)$ obtainable from these theorems improves related results of Milloux [Prace Mat.-Fiz. 41, 39–54 (1934)] and of Wiman [Acta Math. 66, 123–145 (1936)] by being more general and by giving better appraisals for the error terms. *W. Wasow* (Los Angeles, Calif.).

Manacorda, Tristano. Sul comportamento asintotico degli integrali di una classe di equazioni differenziali non lineari. Boll. Un. Mat. Ital. (3) 7, 137–142 (1952).

It is shown that under appropriate hypotheses all solutions of the differential equation $\ddot{x}(t)+x(t)=\phi(t)+f(t)g(x)$ tend to solutions of the linear differential equation

$$\ddot{x}(t)+x(t)=\phi(t),$$

as $t\to\infty$. The most important of these hypotheses are: (a) the functions $g(x)$ and dg/dx satisfy a uniform Lipschitz condition for all x; (b) the derivative dg/dx is uniformly bounded for all x; (c) the integral $\int_t^\infty dv\left|\int_v^\infty f(s)ds\right|$ exists; (d) all solutions of $\ddot{x}+x=\phi$ are bounded, for $t\geqq t_0$. For the proof the differential equation is replaced by a Volterra integral equation whose solution is appraised by the method of successive approximations. *W. Wasow.*

Malkin, I. G. On a theorem concerning stability of motion. Doklady Akad. Nauk SSSR (N.S.) 84, 877–878 (1952). (Russian)

Consider systems

$$(1)\qquad \dot{x}_s=X_s(t,x_1,\cdots,x_n)+X_s'(t,x_1,\cdots,x_n);$$

and (2) $\dot{x}_s=X_s$; the X_s are assumed to be periodic in t and the X_s, X_s' sufficiently regular to insure the existence and uniqueness of the solutions in the region $t\geqq 0$, $|x_s|\leqq H$; $x_s=0$ is assumed to be a solution of both systems and $X_s'(t,x_1,\cdots,x_n)\to 0$ uniformly as $t\to\infty$. Then, if $x_s=0$ is an asymptotically stable solution of (2), the same is true of (1). *J. L. Massera* (Montevideo).

Solncev, Yu. K. On stability according to Lyapunov of the equilibrium positions of a system of two differential equations in the case of discontinuous right hand sides. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 144–180 (1951). (Russian)

Let G be a planar region subdivided by a finite graph Γ (which includes the boundary of G and whose arcs are differentiable) into subregions $G_1,\cdots,G_N$. Let there be given a system

$$(1)\qquad \dot{x}=X(x,y),\quad \dot{y}=Y(x,y)$$

where X, Y in each $\bar{G}_\nu$ are continuous and in each compact subset of $\bar{G}_\nu$ satisfy a Lipschitz condition. A boundary point q of one of the G_ν may belong to one of four types. Denote by V the vector (X, Y) and by γ a path. Then: (a) for some ν, $V(p)\to 0$ as $p\to q$ on G; (b) no γ on any G_ν leaves G_ν at q; (c) no γ on any G_ν enters G_ν at q; (d) the two circumstances excluded under (b), (c) both arise at q. Under (b) one agrees that γ for all t after t_0 when it reaches p coincides with q and under (c) the same but for all t before t_0 when it reaches q. A boundary point q is quasistationary (=q.s.) whenever all the boundary points nearer to q than some ϵ are of type (a), (b) or else points common to the boundaries of just two regions G_μ, G_ν with both vectors $V(q)$ not tangent to the boundary arc through q.

The author discusses more especially the stability of the q.s. points. In this connection a basic role is played by the reduced systems. The graph Γ for such a system consists of a finite number of rays issued from the origin 0 and the G_ν are the consecutive sectors determined by the rays. In G_ν (1) has the form

$$(2)\qquad \dot{x}=a_{1\nu}x+b_{1\nu}y,\quad \dot{y}=a_{2\nu}x+b_{2\nu}y$$

with constant coefficients and nonzero determinant of the coefficients. Moreover the boundaries are regular in the following sense: along the sector boundary, say of G_ν, the vector V is never tangent to the boundary. There are two types of sectors: (a) sectors with rotation, i.e., such that the paths entering from one side leave from the other; (b) sectors with screen, i.e., those for which no path behaves in the preceding manner.

A necessary and sufficient condition for the origin to be stable relative to a reduced system with at least one sector with screen is that it be stable relative to every sector with screen. The stability is then asymptotic. A complicated analytical stability criterium is also given when the sectors are all with rotation.

Similar systems are considered save that the boundaries of the sectors are curvilinear and not paths. If they are not tangent, one replaces each sector by the sector of the tangents. The stability of the resulting reduced system governs that of the given system. If some of the boundaries are tangent, one suppresses all but the first of a tangent set and proceeds in the same manner. Extension is also made to systems (1) with the G_ν still curvilinear sectors and X, Y non-linear with first degree terms suitably restricted.

The author considers next a general, bounded G_ν with system (1) linear in each G_ν. Here again the same results are obtained for isolated q.s. points under appropriate restrictions on the individual linear systems. Finally, stability is considered for a set of q.s. points making up a tree T in the graph Γ. The general method consists in attaching a suitable reduced system to each vertex of T. The stability of every one of these reduced systems governs the stability of T. [References: Andronov and Maĭer, Avtomatika i Telemehanika 8, 314–334 (1947); these Rev. 12, 413; Nemyckiĭ and Stepanov, Qualitative theory of differential equations (Russian), Moscow-Leningrad, 1947; these Rev. 10, 612; E. A. Barbašin, Moskov. Gos. Univ. Učenye Zapiski 135, Matematika 2, 110–133 (1948); these Rev. 11, 443]. *S. Lefschetz* (Princeton, N. J.).

Gavrilov, N. I. On stability according to Lyapunov of systems of linear differential equations. Doklady Akad. Nauk SSSR (N.S.) 84, 425–428 (1952). (Russian)

Consider $\dot{x}_i=\sum_{k=1}^n P_{ik}(t)x_k$; let

$$u_i(t)=\exp\left[\int_T^t \operatorname{Re} P_{ii}(t)dt\right],$$

$$b_{ii}=1,\quad b_{ik}(t)=-u_i(t)\int_T^t\{|P_{ik}(t)|/u_i(t)\}dt\quad (i\neq k);$$

let $D_n(t_1,\cdots,t_n)$ be the determinant with elements $b_{ik}(t_i)$. Theorem: If there is a T such that for any choice of $t_i\geqq T$: (a) $D_n(t_1,\cdots,t_n)>0$, all diagonal minors of D_n being $\geqq 0$; (b) functions u_iA_{ik}/D_n are bounded, where A_{ik} is the algebraic complement of the element (i, k) in D_n; then the solution $x=0$ is stable. It is shown that this "determinantal"

criterion is stronger than all criteria previously given by several authors. No proofs are given. *J. L. Massera.*

Gavrilov, N. I. On a method in the theory of stability according to Lyapunov. Doklady Akad. Nauk SSSR (N.S.) **84**, 657–660 (1952). (Russian)

The system considered in the article reviewed above is studied by the "method of the generalized characteristic equation", i.e., the consideration of the roots $\lambda_s(t)$ of $|P_{ik}(t)-\lambda\delta_{ik}|=0$. The application of the "determinantal" criterion gives a new sufficient condition for stability weaker than Persidskiĭ's [Izvestiya Fiz.-Mat. Obšč. i Naučno-Issled. Inst. Mat. Meh. Kazan. Univ. (3) 11, 29–45 (1938)]; the precise statement is too lengthy to be given here in extenso. *J. L. Massera* (Montevideo).

Haacke, Wolfhart. Über die Stabilität eines Systems von gewöhnlichen linearen Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten, die von Parametern abhängen. I. Math. Z. **56**, 65–79 (1952).

The system of differential equations under consideration is of the form $y''+\lambda^2R(x)y+\lambda A(x,\lambda)y=0$. Here y is an n-dimensional column vector, λ is a large positive parameter, and $R(x)$ and $A(x,\lambda)$ are square complex-valued matrix functions of order n with period 2π in x. It is further assumed that $A(x,\lambda)$ possesses an asymptotic expansion of the form $A(x,\lambda)\sim\sum_{\rho=0}^{\infty}\lambda^{-\rho}A_\rho(x)$, and that $R(x)=\{\delta_{\mu\nu}r_\mu^2(x)\}$, with $r_\mu^2\not\equiv r_\nu^2$ for $\mu\neq\nu$. The system is called stable if all its solutions are uniformly bounded on the whole real x-axis, i.e., if its characteristic roots σ_τ $(\tau=1,\cdots,2n)$ are all distinct and of modulus 1. An asymptotic form, for large λ, of the characteristic equation whose roots are σ_τ is calculated by means of explicit asymptotic expressions for a set of $2n$ independent solutions of the differential system. These solutions are found by using a result of Perron [S.-B. Heidelberger Akad. Wiss. **9A**, nos. 13, 15 (1918); **10A**, no. 6 (1919)]. The generally highly restrictive stability condition is studied in further detail in the special case that $R(x)$ and $A(x,\lambda)$ are real and, in addition, even functions with respect to a point $x=a$. It is shown first that the system is unstable for all sufficiently large λ, if $r_\mu^2<0$ for some μ. If all r_μ^2 are positive, the system turns out to be stable except for certain explicitly calculated λ-intervals of length $O(\lambda^{-1/2})$.

W. Wasow (Los Angeles, Calif.).

Hayashi, Chihiro. Stability investigation of the non-linear periodic oscillations. Mem. Fac. Eng. Kyoto Univ. **14**, 92–102 (1952).

The author investigates the stability of periodic solutions of $d^2v/dt^2=F(v,dv/dt,t)$, F having period T in t, by a formal consideration of the variational equation (of Hill's type). The reviewer cannot follow the author's arguments and statements of results, nor his critical comments on previous work by L. Mandelstam and N. Papalexi [Z. Physik **73**, 223–248 (1931)]. *G. E. H. Reuter* (Manchester).

Vinograd, R. Ê. Some criteria of boundedness of the solutions of a system of two linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **85**, 265–268 (1952). (Russian)

Under rather general conditions the author shows the system can be put into a canonical form $x_1'=\lambda(t)x_1-\omega(t)x_2$ and $x_2'=\omega(t)x_1-\mu(t)x_2$ by means of an orthogonal transformation of the dependent variables. It is assumed $\lambda(t)>0$, $\mu(t)>0$, $\int_0^\infty\lambda(t)dt=\int_0^\infty\mu(t)dt=\infty$. Let $J=\int_0^t(\lambda(\tau)-\mu(\tau))d\tau$ and $\limsup J<\infty$ as $t\to\infty$. Let $2\omega/(\lambda+\mu)=h(t)$. Let $h(t)\geqq 1+\epsilon$ for some $\epsilon>0$ and let $h'(t)$ have a fixed sign. Then the solutions of the system are bounded as $t\to\infty$. If $J(t)\to-\infty$ as $t\to\infty$, the solutions tend to zero. Many other results are given. *N. Levinson* (Cambridge, Mass.).

Volosov, V. M. Nonlinear differential equations of the second order with a small parameter with the highest derivative. Mat. Sbornik N.S. **30(72)**, 245–270 (1952). (Russian)

Consider the equation $\mu y''+Q(x,y)=0$, where $\mu\to 0$; Q has continuous bounded second derivatives in $[x_0,\bar{x}]$, $|y|<\infty$, $Q[x,f(x)]\equiv 0$ where $f(x)$ has a continuous second derivative, $Q(x,y)\gtrless 0$ when

$$y\gtrless f(x),\quad m|y-f(x)|\leqq Q(x,y)\leqq M|y-f(x)|,\quad 0<m<M.$$

The following facts are proved: a) the solution through $[x_0,f(x_0)]$ tends uniformly to $y=f(x)$ as $\mu\to 0$; b) the solution $y(x,u_0,\mu)$ through $[x_0,f(x_0)+u_0]$, $u_0\neq 0$, oscillates, as $\mu\to 0$, about $y=f(x)$, the number of these oscillations in any fixed interval increasing indefinitely; c) there are two functions $y=F_i(x)$, $i=1,2$, $F_1<f<F_2$, which are solutions of a certain system of differential equations (independent of μ, u_0) with initial conditions depending on u_0, such that given $\epsilon>0$ there is a $\delta>0$ such that if $\mu<\delta$, the maxima and minima of the solution $y(x,u_0,\mu)$, $u_0\neq 0$, lie in the neighborhoods of width ϵ of the curves $y=F_i(x)$; d) given any smooth function $\psi(x,y)$, the mean value $\lim_{\mu\to 0}\int_a^b\psi[x,y(x,u_0,\mu)]dx$ may be explicitly calculated in terms of Q, f, F_i.

J. L. Massera (Montevideo).

Smith, R. A. On an equation connected with the theory of triode oscillations. Proc. Cambridge Philos. Soc. **48**, 698–717 (1952).

The differential equation $\ddot{x}+k\dot{x}f(x)+x=pk\lambda\cos(\lambda t+\alpha)$ is investigated for existence, amplitude, and stability of periodic solutions when the parameter k is small. The method resembles that of M. L. Cartwright [Contributions to the theory of nonlinear oscillations, Princeton, 1950, pp. 149–241; these Rev. 11, 722]. The principal aim of the author is to establish conditions on $f(x)$ which guarantee that the results obtained in the paper cited above for the case $f(x)=x^2-1$ remain qualitatively valid. The function $f(x)$ is assumed to satisfy the following conditions: (1) $f'''(x)$ is continuous; (2) $f''(x)\geqq 0$; (3) $|f'(x)|\to\infty$ as $|x|\to\infty$; (4) $f(0)<0$. The main criterion depends on properties of the function

$$\theta(b)=\frac{2}{\pi}\int_0^{\pi/2}\sin\tau\int_0^{b\sin\tau}\{f(x)+f(-x)\}dxd\tau.$$

Let θ_1 and θ_2 be the positive roots of $\theta'(b)$ and $\theta(b)$, respectively, and assume that $b\theta''(b)\geqq\theta'''(b)$. Then the number and nature of solutions of the differential equation having period $2\pi/\lambda$, with λ near 1, is shown to vary with λ in qualitatively the same way as in the case of Van der Pol's equation. This criterion is used to answer the special question as to how "flat" $f(x)$ could be without destroying the essential similarity to the case $f(x)=x^2-1$. The investigations are then extended to the subharmonics of order $\frac{1}{2}$.

W. Wasow (Los Angeles, Calif.).

Gavurin, M. K. On systems of differential equations of the form $y'=Ay^2-2By+C$. Doklady Akad. Nauk SSSR (N.S.) **84**, 205–208 (1952). (Russian)

Consider a system

$$\dot{y}^i=\sum_{j,k=1}^{n}a^i_{jk}y^jy^k-2\sum_{l=1}^{n}b_l^iy^l+c^i,\quad y^i(0)=\eta_i,$$

$\dot{y}$ the derivative with respect to the complex variable t, $a^i_{jk}=a^i_{kj}$, $b_i{}^i$, c^i constants. Let $A=(a^i_{jk})$ (three-dimensional matrix), $B=(b_i{}^i)$, $C=(c^i)$, $y_0=(\eta^i)$, $x, y, \cdots$ represent similar column-vectors. The (two dimensional) matrix $(\sum_{k=1}^n a^i_{jk}x^k)$ will be written Ax and $Axz=(Ax)z=Axx$, so that the original system may be written $\dot{y}=Ay^2-2By+C$. Consider the set $\mathfrak{M}$ of all matrices of the form Ax; assume $\mathfrak{M}$ is a (commutative) semi-group containing the identity $I=Az_0$, the matrix B and the matrix $(B^2-AC)^{1/2}$ (supposed to exist). Then, if $y_1, y_2=(B\pm(B^2-AC)^{1/2})z_0$ (these are the "roots" of $Ay^2-2By+C=0$), $K_1=A(y_0-y_1)$, $K_2=A(y_0-y_2)$, $K=K_1-K_2$, $P=e^{tK}K_1$, the solution of the differential equations is given by $y=(K_2-P)^{-1}(K_2y_1-Py_2)$.

J. L. Massera (Montevideo).

Montaldo, Oscar. Sull'integrazione dei sistemi di Riccati. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 47–58 (1952).

For a system of n ordinary linear differential equations of order 2 and of rank r in the second derivatives of the n unknown functions ($1<r\leqq n$) this paper gives a set of invariants (generalizing that given by Chiellini [same Rend. **18**, 44–58 (1949); these Rev. **11**, 665] for $n=2$) whose constancy is a necessary and sufficient condition for the equivalence of the system to one with constant coefficients. This result is further applied to determine all systems of Riccati equivalent to a linear system with constant coefficients.

J. M. Thomas (Durham, N. C.).

Stöhr, Alfred. Über die Differentialgleichungen eines dynamischen Weltmodells. II. Math. Nachr. **7**, 339–357 (1952).

In part I [Math. Nachr. **6**, 71–88 (1951); these Rev. **13**, 746] the author studied the general properties of a system of differential equations which has applications to some cosmological models. In this paper the author obtains explicit solutions for special cases. *A. Schild.*

Herrmann, A. Eine Anwendung der Poincaré'schen Formel des Matrizencalcüls. Ann. Univ. Saraviensis **1**, 93–101 (1952).

Let $p(s)$ denote a polynomial of degree n and let $h(s)$ be one of lower degree. From elementary properties of the Laplace transformation it follows that h/p is the transform of a function $Y(t)$ that satisfies a linear homogeneous differential equation of order n whose coefficients are those in $p(s)$ provided that the initial values $Y^{(i)}(0)$ $(i=0, 1, \cdots, n-1)$ are properly chosen in terms of the coefficients in the two polynomials p and h. The author writes the differential equation as one of the first order in matrices and presents a discussion of the above fact with the aid of some matrix theory. Misprints are numerous. *R. V. Churchill.*

Naĭmark, M. A. On the deficiency index of linear differential operators. Doklady Akad. Nauk SSSR (N.S.) **82**, 517–520 (1952). (Russian)

The formal linear differential operator

$$l(y)=(-1)^n(p_0y^{(n)})^{(n)}+(-1)^{n-1}(p_1y^{(n-1)})^{(n-1)}+\cdots+p_ny$$

is considered, where $1/p_0, p_1, \cdots, p_n$ are real, measurable functions on $0<x<+\infty$, and summable on each finite interval $0\leqq x\leqq a$, $a>0$. Since l coincides with its Lagrange adjoint, l gives rise, in a natural way, to a closed symmetric operator L defined on a dense subset of the Hilbert space $L^2(0, \infty)$. Let D_0 denote the set of all functions $y\in L^2(0, \infty)$, vanishing outside closed bounded intervals $\epsilon\leqq x\leqq a$ $(0<\epsilon<a)$, for which $l(y)$ makes sense and $l(y)\in L^2(0, \infty)$. Define L_0 to be the operator with domain D_0, and $L_0(y)=l(y)$, for $y\in D_0$. Then L is the closure of L_0. Since the coefficients in l are real, the deficiency index of L has the form (m, m). It is known [Glazman, same Doklady (N.S.) **64**, 151–154 (1949); these Rev. **10**, 538; see also Kodaira, Amer. J. Math. **72**, 502–544 (1950), esp. pp. 509, 518; these Rev. **12**, 103] that $n\leqq m\leqq 2n$ in the above case. The author states (with no proof) six sufficient conditions on the p_i in order that the deficiency index of L be (n, n), or $(n+1, n+1)$. An example is: if there exist constants $a_0\neq 0, a_1, \cdots, a_n$ such that the functions $(1/p_0)-(1/a_0)$, $p_1-a_1, \cdots, p_n-a_n$ are summable on $0<x<\infty$, then the deficiency index of L is (n, n). It is stated that the proofs make use of results on the asymptotic behavior of the solutions of $l(y)=\lambda y$, as $x\to\infty$. *E. A. Coddington.*

Kreĭn, M. G. On the application of an algebraic proposition in the theory of matrices of monodromy. Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 171–177 (1951). (Russian)

Let $\dot{x}=A(t)x$, A a complex periodic matrix (period ω), x a complex n-vector, $x=U(t)x_0$ a general solution ($U(0)=I$); the characteristic roots of $U(\omega)$ are called multipliers. Let $\mathfrak{P}$ be the set of all periodic integrable matrices $H(t)$ such that $(H(t)x, x)>0$ for all x and almost all t; $\bar{\mathfrak{P}}$ is the set of matrices such that $(H(t)x, x)$ is $\geqq 0$, its integral in $(0, \omega)$ being >0. 1) If a non-singular hermitian constant matrix G exists having signature $p-q$, such that $H=GA+A^*G\in\mathfrak{P}$, then p (q) multipliers are numerically greater (smaller) than 1. 2) If $P(t)$ is continuous and $\operatorname{Im}(P(t)x, x)<0$ for all t, then n multipliers of $\dot{x}+P(t)x=0$ are numerically greater and n numerically smaller than 1. 3) If a non-singular hermitian constant matrix G exists having signature $p-q$, such that $GA=-iH$, $H\in\bar{\mathfrak{P}}$, then for real λ all multipliers of $\dot{x}=\lambda Ax$ not lying on the unit circle appear in pairs symmetric with respect to the unit circle, the number of such pairs being $\leqq\inf(p, q)$; if $\operatorname{Im}\lambda>0$ (<0), then p (q) multipliers lie inside and q (p) outside the unit circle. 4) If $P\in\bar{\mathfrak{P}}$, all statements of Theorem 3 are true for the system $\dot{x}+\lambda^2Px=0$, with $p=q=n$. *J. L. Massera* (Montevideo).

Checcucci, Vittorio. Funzioni olomorfe di più matrici e sistemi di equazioni lineari del primo ordine ai differenziali totali. Rivista Mat. Univ. Parma **2**, 375–382 (1951).

Let $\mathfrak{A}$ be an algebra of matrices of order p over the complex field and $\mathfrak{C}$ its center. A matrix y is a holomorphic function of matrices $x_1, \cdots, x_n$, restricted to certain open sets $J_1, \cdots, J_n$ of $\mathfrak{C}$, if its coordinates (in $\mathfrak{A}$) are holomorphic functions of the coordinates of $x_1, \cdots, x_n$ (in $\mathfrak{C}$). As in the scalar theory, derivatives exist and the order of differentiation may be interchanged. An expression $\sum_{r=1}^n A^{(r)}dx_r$, with the $A^{(r)}$ holomorphic functions of the x_r, $dx_r=x'_r-x_r$ and x_r and x'_r in $\mathfrak{C}$, is an exact differential dy if and only if $\partial A^{(r)}/\partial x_s=\partial A^{(s)}/\partial x_r$, and then $y=\int_{x^0}dy$, the integral being interpreted in much the usual sense. If $A^{(r)}A^{(s)}=A^{(s)}A^{(r)}$, then $(\sum_{r=1}^n A^{(r)}dx_r)z$ is an exact differential dz with solution $z=(I+S_1+S_2+\cdots)c$, where $S_1=y$, $S_{r+1}=\int_{x^0}dy\cdot S_r$ and $z=c$ at x^0. When $n=2$, $dz=Adx+Bdy$ is solved by introducing a Jordan canonical form $H^{-1}AH=D+J$ for A and a related form $K^{-1}(H^{-1}BH)K=D'+J'$ for B. Then $z=He^{xJ}Ke^{yJ'}e^{xD+yD'}Q'$ where D and D' are diagonal, J and J' are nilpotent (so e^{xJ} and $e^{yJ'}$ reduce to polynomials) and Q' is an arbitrary nonsingular matrix. The author claims only to have presented in the appropriate matrix language

results essentially already known [cf. B. Pini, Boll. Un. Mat. Ital. (3) 5, 255–264 (1950); these Rev. 12, 708].
W. Givens (Knoxville, Tenn.).

Hartman, Philip. On the zeros of solutions of second order linear differential equations. J. London Math. Soc. 27, 492–496 (1952).

The author considers the differential equation (1) $y''+[\lambda-q(x)]y=0$, where $q(x)$ is continuous for $x\geqq 0$ and $q(x)\to\infty$ as $x\to\infty$. He extends a theorem of Titchmarsh by showing that if q is a continuous, increasing, convex function, the number $N(\lambda)$ of zeros of a nontrivial solution $y(x)$ of (1) satisfies the condition $\pi N(\lambda)=\int_0^{\varphi(\lambda)}(\lambda-q)^{\frac{1}{2}}dx+O(1)$ as $\lambda\to\infty$. Here $q(\varphi(\lambda))=\lambda$. *W. Leighton.*

Mohr, Ernst. Über das Sturm-Liouvillesche Eigenwertproblem. Math. Nachr. 7, 305–322 (1952).

The author studies the Sturm-Liouville system

$$(1)\qquad -(py')'+(q-\lambda r)y=f,\quad \alpha y(a)+\beta y'(a)=0,\quad \gamma y(b)+\delta y'(b)=0,$$

where p, q, r are real continuous functions of the real variable x, and p and r are positive on the interval $a\leqq x\leqq b$. The function $f(x)$ is assumed to be integrable Riemann in the interval, and $\alpha^2+\beta^2>0$, $\gamma^2+\delta^2>0$. To solve the given differential system the author considers a corresponding difference equation and applies variational methods to it. Typical of the treatment is the following. Let the interval $[0, 1]$ be divided by the partition $0=x_0<x_1<\cdots<x_n=1$ into n equal parts of length h, and denote by p_i, q_i, r_i, y_i, respectively, the values of the functions p, q, r, y at $x=x_i$. It is then required that

$$\sum_{i=0}^{n-1}hp_i\left(\frac{\Delta y_i}{h}\right)^2+\sum_{i=1}^{n-1}q_iy_i^2-\frac{\alpha}{\beta}p_0y_0^2+\frac{\gamma}{\delta}p_{n-1}y_n^2$$

be stationary subject to the boundary condition

$$\sum_{i=1}^{n-1}hr_iy_i^2=1.$$

The management of the resulting algebraic complications becomes a tour de force from which emerge the desired theorems. *W. Leighton* (St. Louis, Mo.).

Seifert, George. A third order irregular boundary value problem and the associated series. Pacific J. Math. 2, 395–406 (1952).

The author continues the work on the boundary value problem

$$u'''(x)+p(x)u'(x)+\{q(x)+\lambda\}u(x)=0$$

with $u(0)=u'(0)=u''(1)=0$, begun in an earlier paper [Quart. Appl. Math. 9, 210–218 (1951); these Rev. 13, 655]. The coefficients are of the form $p(x)=x\psi_1(x^3)$, $q(x)=\psi_2(x^3)$, where $\psi_1(z)$ and $\psi_2(z)$ are real for real z and analytic in $|z|\leqq 1$. Questions were raised at the Second International Aeronautics Congress held in New York in 1949 concerning the conditions under which a function $f(x)$ can be expanded in terms of the associated characteristic functions. It is shown here that if $f(x)$ is of the form $x^2\phi(x^3)$ where $\phi(z)$ is analytic in $|z|\leqq 1$, the formal series for $f(x)$ converges uniformly to $f(x)$ on $0\leqq x\leqq 1$. *E. T. Copson.*

Coddington, E. A. On the spectral representation of ordinary self-adjoint differential operators. Proc. Nat. Acad. Sci. U. S. A. 38, 732–737 (1952).

Let in the formally self-adjoint operator

$$L=p_0(d/dx)^n+p_1(d/dx)^{n-1}+\cdots+p_n$$
$$=(-1)^n(d/dx)^n(\bar{p}_0\cdot)+(-1)^{n-1}(d/dx)^{n-1}(\bar{p}_1\cdot)+\cdots+\bar{p}_n,$$

the coefficients p_k be complex-valued functions with $(n-k)$ continuous derivatives on an open real interval (a, b), and $p_0(x)\neq 0$ on (a, b). Extending the results due to Weyl, Stone, Titchmarsh, Kodaira (and simplified proofs by Levinson, reviewer and Levitan) concerning the case of $n=2$ and real coefficients p, the author considers a self-adjoint boundary-value problem on (a, b) with the operator L. Let D be the set of all $u\in L_2(a, b)$ such that $Lu\in L_2(a, b)$, and let D_0 be the set of all $u\in D$ such that

$$\lim_{a'\to a,\,b'\to b}\int_{a'}^{b'}(\bar{v}Lu-u\overline{Lv})dx=0\quad\text{for all } v\in D.$$

Let T_0 be the operator L restricted to the domain D_0. Assuming T_0 to be self-adjoint (this corresponds to the situation where L is of limit point type at a and b), the author proves, for the spectral resolution $T_0=\int\lambda dE(\lambda)$, the concrete representation

$$E(\Delta)u(x)=\int_a^b E(x, y, \Delta)u(y)dy,\quad =[c, d),$$

$$E(x, y, \Delta)=\int_\Delta\sum_{j,k=1}^{n}s_j(x, \lambda)\overline{s_k(y, \lambda)}d\rho_{jk}(\lambda),$$

with hermitian, positive semi-definite matrix $(\rho_{jk}(\Delta))$ and n linearly independent solutions $s_j(x, \lambda)$ of $Ls_j=\lambda s_j$. The proof is obtained by a natural limiting process from the case associated with the case of a finite closed interval $[a', b']$, $a<a'<b'<b$. *K. Yosida* (Nagoya).

Olsson, P. O. A differential equation for the phase shifts in scattering problems. Ark. Fys. 4, 217–221 (1952).

In order to find an equation for the phase shifts the phase has to be defined for finite values of the argument since it is usually defined only asymptotically. A differential equation of the first order for the phase is given and useful results presented. *N. Levinson* (Cambridge, Mass.).

Birindelli, Carlo. Integrazione dei sistemi lineari ai differenziali totali illimitatamente integrabili in due variabili in un prescritto campo semplicemente connesso del piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 518–523 (1952).

Soit le système aux différentielles totales (à p fonctions inconnues x_i de deux variables indépendantes t_1, t_2)

$$dx_i=\left[g_i^{(1)}(t)+\sum_{k=1}^{k=p}x_k\phi_i^{1,k}(t)\right]dt_1+\left[g_i^{(2)}(t)+\sum_{k=1}^{k=p}x_k\phi_i^{2,k}(t)\right]dt_2,$$

(t) représentant (t_1, t_2), les g, ϕ étant continues, ainsi que leurs dérivées premières, dans une aire simplement connexe T du plan (t_1, t_2). Sous la seule condition que les conditions d'intégrabilité complète soient vérifiées identiquement dans T, l'auteur démontre l'existence, dans tout le champ donné T, de la solution prenant pour (t_1^0, t_2^0) un système de valeurs arbitrairement données $(x_1^0, x_2^0, \cdots, x_n^0)$. *M. Janet.*

Avakumović, Vojislav G. Über die Eigenfunktionen der Schwingungsgleichung. Acad. Serbe Sci. Publ. Inst. Math. 4, 95–96 (1952).

For an eigenvalue problem $\Delta U+\lambda U=0$ in a domain S with $U=0$ on the (sufficiently smooth) boundary of S, Carleman, and later Minakshisundaram, have established the estimates

$$\sum_{\lambda_n\leqq\lambda}\phi_n^2(P)=C_k\lambda^{k/2}+o(\lambda^{k/2})$$

and, for $P\neq Q$,

$$\sum_{\lambda_n\leqq\lambda}\phi_n(P)\phi_n(Q)=o(\lambda^{k/2}),$$

where C_k is a well-determined constant. Now, the author claims, and very briefly proves by the use of a one-sided Tauberian theorem, that the formulas underlying these estimates permit one to replace the error term $o(\lambda^{k/2})$ by the much sharper one $O(\lambda^{(k-1)/2})$. *S. Bochner.*

Bochner, S. Partial differential equations and analytic continuations. Proc. Nat. Acad. Sci. U. S. A. **38**, 227–230 (1952).

Let $f(x_1, \cdots, x_n)$ be an analytic solution of two partial differential equations with constant coefficients in a domain U. Here the first equation shall be "elliptic semi-homogeneous", i.e., possess a fundamental solution with the property that its derivatives of sufficiently high order are $O(|x|^{-n-1})$ at infinity. The second equation shall not involve derivatives with respect to all of the independent variables. The main theorem of the paper states that if D is a domain with connected boundary B, and U is a neighbourhood of B, then f can be continued analytically from U into all of $D+U$. *F. John* (New York, N. Y.).

Hornich, Hans. Zur Lösbarkeit von gewissen elliptischen Differentialgleichungen. J. Reine Angew. Math. **189**, 204–206 (1952).

The author considers the equation

$$a^2(r^2\partial^2u/\partial r^2+r\partial u/\partial r)+\partial^2u/\partial\varphi^2=f,$$

where a is an irrational real number between 0 and 1,

$$f=\sum_{n=0}^{\infty} r^n \sum_{\substack{\nu=n,\ n-2,\ \cdots \\ \nu\geq 0}} (\alpha_{\nu n}\cos\eta\varphi+\beta_{\nu n}\sin n\varphi)$$

($|\alpha_{\nu n}|\leqq M$, $|\beta_{\nu n}|\leqq M$ for all ν, n). It is shown that a necessary condition that the equation have no twice differentiable solution near $r=0$ is that a belong to the set of Liouville transcendental numbers for which $\liminf |a-\nu/n|^{1/n}=0$ for all pairs of integers ν, n whose difference is even. *F. Browder* (Cambridge, Mass.).

Miranda, Carlo. Sulle proprietà di minimo e di massimo delle soluzioni delle equazioni a derivate parziali lineari del secondo ordine di tipo ellittico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 117–120 (1951).

This paper contains a number of theorems concerning the maximum-minimum principle for solutions of linear elliptic equations in the plane. Stronger forms of these results, however, were established in a direct and elementary manner for elliptic differential equations in an arbitrary number of independent variables by E. Hopf [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1927**, 147–152]. *F. E. Browder.*

Gilbarg, David. The Phragmén-Lindelöf theorem for elliptic partial differential equations. J. Rational Mech. Anal. **1**, 411–417 (1952).

The maximum-minimum principle for elliptic partial differential equations of the second order is extended to unbounded domains for solutions satisfying a growth condition at infinity by use of an analogue of the Phragmén-Lindelöf argument. Let $L(u)=au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y$ be a linear elliptic differential operator defined on a plane region D contained in a half-plane and with coefficients continuous on D. The coefficients of the second order terms are assumed uniformly bounded on D while d and e are uniformly bounded and $O(r^{-1-\epsilon})$ for some $\epsilon>0$ and large r. Assume L to be uniformly elliptic on D. Suppose that $u(z)$ of class C' satisfies the inequality $L(u)\leqq 0$ in D and has a non-negative limit as z approaches any boundary point of D. Suppose further that $\liminf_{r\to\infty} m(r)/r=0$, where $m(r)=\min_{|z|=r} u(z)$. Then $u(z)\geqq 0$ in D. (It might be noted that it follows, further, that $u(z)>0$ in D (connected) unless u is identically zero.) The proof employs the harmonic majorant applied by Nevanlinna in a proof of the classical Phragmén-Lindelöf theorem. The author notes a simple extension of his theorem to elliptic equations which are not necessarily linear and remarks that this extension can be applied to the proof of comparison theorems for compressible flows. *F. E. Browder* (Boston, Mass.).

Hopf, Eberhard. Remarks on the preceding paper by D. Gilbarg. J. Rational Mech. Anal. **1**, 419–424 (1952).

An extension by Gilbarg [see the preceding review] of the maximum-minimum principle of the Phragmén-Lindelöf type for linear elliptic partial differential operators in two independent variables is generalized in this paper to elliptic operators in an arbitrary number of independent variables. *F. E. Browder* (Boston, Mass.).

Višik, M. I. On a general form of solvable boundary problems for homogeneous and nonhomogeneous elliptic differential equations. Doklady Akad. Nauk SSSR (N.S.) **82**, 181–184 (1952). (Russian)

Let D be a domain in Euclidean n-space, Γ its boundary. If Ω_γ is a linear subset of $L_2(D)$, $B(\Gamma)$ a Banach space of functions on Γ, the linear mapping γ of Ω_γ into $B(\Gamma)$ is called a boundary operator if $\gamma f_0=0$ for every f_0 in Ω_γ which vanishes on a neighborhood of Γ. The author considers boundary-value problems asking for (generalized) solutions of the second-order linear elliptic partial differential equation

$$Lf=-\sum_{i,k=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right)+\sum_{i=1}^{n}b_i\frac{\partial f}{\partial x_i}+c(x)f(x)=h(x)$$

with $\gamma f=\varphi$ for $h\in L_2(D)$, $\varphi\in B(\Gamma)$ and $c(x)\geqq 0$ for x in D. Such a boundary-value problem is said to be solvable (completely solvable) if there exists a unique solution f for each pair h, φ and if we set $V(h,\varphi)=f$, the operator V is a continuous (completely continuous) linear transformation from $L_2(D)\times B(\Gamma)$ into $L_2(D)$. Theorem 1 states that a necessary and sufficient condition that the boundary value problem be solvable (completely solvable) is that the boundary operator γ can be put in the form $\gamma f=Q(\gamma_1 f-\bar{C}\gamma_2 f)$ where

$$\gamma_1 f=f|_\Gamma,\quad \gamma_2 f=\partial f/\partial\gamma|_\Gamma-Pf|_\Gamma,$$

$$\partial f/\partial\nu|_\Gamma=\sum_{i,k=1}^{n}a_{ik}\cos(n,x_i)\partial f/\partial x_i|_\Gamma,\quad Pf|_\Gamma=\partial u/\partial\nu|_\Gamma,$$

where u is the solution of the problem $Lu=0$, $u|_\Gamma=f|_\Gamma$. Here $\bar{C}$ and Q are two linear operators satisfying the following conditions. Let H_1 be the image of Ω_γ under γ_1 with the norm of $\gamma_1 f$ equal to the L_2-norm of f, H_2 the image of Ω_γ under γ_2 with a similar norm. Then $\bar{C}$ is required to be a bounded (completely continuous) linear transformation from H_2 to H_1, Q a linear operator with domain in H_1 and range in $B(\Gamma)$ having a continuous (completely continuous) inverse. Theorem 2 considers the boundary-value problem $Lu=0$, $\partial u/\partial\nu|_\Gamma+Au|_\Gamma=\varphi$ where the linear operator A has its domain dense in $L_p(\Gamma)$ ($p\geqq 2n/n-1$) with range in $L_q(\Gamma)$ ($1\leqq q\leqq 2(n-1)/n-2$). It is assumed that this problem has a single solution u in $L_2(D)$ for each φ in $L_q(\Gamma)$ and, setting $V_0\varphi=u$, that V_0 is a bounded (completely continu-

ous) linear transformation from $L_2(\Gamma)$ to $L_2(D)$. Then the boundary-value problem $Lf=h$, $\partial f/\partial\nu|_\Gamma+Af|_\Gamma=0$ has one and only one solution f for each h in $L_2(D)$ and the operator L^{-1} in $L_2(D)$ is bounded (completely continuous).

The indications given of the proofs depend in an essential fashion upon results of the author which have been published elsewhere. The theorems are extensions of results announced by the author in two previous notes [same Doklady (N.S.) **65**, 785–788 (1949); **77**, 373–375 (1951); these Rev. **11**, 39; **12**, 830]. *F. Browder* (Boston, Mass.).

Oleĭnik, O. A. On properties of solutions of certain boundary problems for equations of elliptic type. Mat. Sbornik N.S. **30(72)**, 695–702 (1952). (Russian)

This paper is concerned with the uniqueness of solutions of mixed boundary-value problems for second-order linear elliptic equations in n independent variables as well as the continuous dependence of solutions on inhomogeneous parts, coefficients, and boundary data. All of the proofs are completely elementary in character. Let

$$L(u)=\sum_{i,k}a_{ik}\frac{\partial^2 u}{\partial x_i\partial x_k}+\sum_i b_i\frac{\partial u}{\partial x_i}+cu=f$$

be the linear elliptic second-order differential equation with coefficients defined and continuous on $\bar{D}$, the closure of the bounded domain D of Euclidean n-space. It is assumed that $c\leqq 0$ in D, L uniformly elliptic on D, the boundary of D has a tangent plane at each of its points, and for each point P of the boundary of D there exists a sphere with interior in D and boundary containing P. The author's Theorem 1, on which all of the subsequent theorems are based, asserts that if u is a non-constant solution of the equation $L(u)=f$ in D which is continuous on $\bar{D}$, $f\leqq 0$, and if P_1 is a boundary point of D at which u assumes a non-positive minimum, then $\lim\inf_{P_2\to P_1}[u(P_2)-u(P_1)]/r(P_2, P_1)>0$ for P_2 taken along a fixed line segment h making an acute angle with the interior normal and $r(P_2, P_1)$ the distance between P_2 and P_1. (Cf. the review of the paper of E. Hopf below.)

The author's Theorem 2 is a simple corollary of Theorem 1. Consider the boundary value problem

$$l(u)=a(P)\partial u/\partial\gamma+b(P)u=\varphi(P)$$

where $a(P)$ and $b(P)$ are finite at each boundary point P, $\partial u/\partial\gamma$ is the directional derivative taken along a given line segment through P making an acute angle with the interior normal, $a^2+b^2\neq 0$ for any P, $a(P)\geqq 0$, $b(P)\leqq 0$. Then if either $c\not\equiv 0$ or $b\not\equiv 0$, the equation $L(u)=f$ with $l(u)=\varphi$ has at most one solution for each pair f, φ. If $c\equiv 0$, $b\equiv 0$, the solutions may differ at most by a constant. For let u be the difference between two solutions. Then $L(u)=0$, $l(u)=0$. Suppose u non-constant. Let P_1 be the boundary point at which u assumes its minimum. At P_1, $l(u)$ will be negative if that minimum is negative. Thus $u\geqq 0$ and similarly $u\leqq 0$.

The concluding part of the paper applies Theorem 1 to estimating the uniform norm of the solution u of $Lu=f$, $l(u)=\varphi$ in terms of the uniform norms of f, φ, and the coefficients of L and l, at least when either c or b is bounded away from zero. Theorem 6 which goes beyond these limitations, at the expense of differentiability assumptions, asserts that if $c\not\equiv 0$ and if the coefficients of L are three times continuously differentiable on $\bar{D}$, then there exists $C_1>0$ not depending on f and φ such that the solution of $L(u)=f$, $l(u)=\varphi$ satisfies the inequality $\max_D u\leqq C_1(\max_D f+\max\varphi)$. Theorem 7 asserts the continuity of the solution in f, φ, and the coefficients of L and l at a solution u for an equation $L(u)=f$ whose coefficients are three times continuously differentiable in $\bar{D}$ and for approximating equations L_1 for which $c_1<0$. *F. Browder* (Boston, Mass.).

Hopf, Eberhard. A remark on linear elliptic differential equations of second order. Proc. Amer. Math. Soc. **3**, 791–793 (1952).

Let R be an open subset of Euclidean n-space, x^0 a point of the boundary of R such that R contains the interior of an n-sphere with x^0 on its boundary. Let L be the linear differential operator of second order on R with no zero order term, $L=\sum_{i,k}a_{ik}(x)\partial^2 u/\partial x_i\partial x_k+\sum_j b_j\partial u/\partial x_j$. Suppose that L is uniformly elliptic on the union of R and x^0 and that all its coefficients are continuous on the same set. Suppose that u is of class C'' in R, $u\geqq 0$, $Lu\leqq 0$ in R. If the limit value of u at x^0 is zero and $u\not\equiv 0$ in R, then it is shown that the normal derivative du/dn at x^0, or properly speaking the lim inf of $\Delta u/\Delta n$, is >0. The proof is elementary. (Cf. the preceding review.) *F. Browder* (Boston, Mass.).

Ladyženskaya, O. On the closure of the elliptic operator. Doklady Akad. Nauk SSSR (N.S.) **79**, 723–725 (1951). (Russian)

Let $\mathfrak{L}$ be the set of all real-valued functions which are twice continuously differentiable on the closure $\bar{\Omega}$ of a bounded open set Ω in $(x_1, \cdots, x_n)$-space, and let W denote the completion of $\mathfrak{L}$ with respect to the norm

$$|u|_W=\left\{\int_\Omega\left[u^2+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2+\sum_{i,j=1}^n\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2\right]d\Omega\right\}^{\frac{1}{2}}.$$

Under suitable hypotheses concerning the elliptic operator L:

$$Lu=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n a_i(x)\frac{\partial u}{\partial x_i}+a(x)u,$$

(thought of as defined on W with values in H, the space of all square-integrable functions on Ω) it is shown that L is closed and one-to-one. A generalization to the operator L of a theorem proved previously [Ladyženskaya, same Doklady (N.S.) **75**, 765–768 (1950); these Rev. **12**, 615] is also proved. *J. B. Diaz* (College Park, Md.).

Ladyženskaya, O. A. Fourier's method for hyperbolic equations. Doklady Akad. Nauk SSSR (N.S.) **74**, 417–420 (1950). (Russian)

A mixed initial-boundary value problem is treated for the equation

$$Lu=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial u}{\partial x_j}\right)-cu=\frac{\partial^2 u}{\partial t^2},\quad c(X)\geqq 0,$$

where $X=(x_1, \cdots, x_n)$ and Lu is an elliptic operator. Let Ω be a domain in X-space with smooth boundary Γ. The given data are $u|_{t=0}=\varphi_0(X)$, $u_t|_{t=0}=\varphi_1(X)$, $u|_\Gamma=0$ for $t\geqq 0$. It is assumed that the a_{ij} have continuous derivatives of order $[n/2]+3$, $c(X)$ of order $[n/2]+2$, φ_0 and φ_1 of order $4[n/4]+6$. The method consists of separating time and space variables, solving the corresponding elliptic eigenvalue problem, and then expanding in a Fourier series. Estimates are obtained to establish convergence of the series.

M. H. Protter (Princeton, N. J.).

Bicadze, A. V. On the general problem of mixed type. Doklady Akad. Nauk SSSR (N.S.) **78**, 621–624 (1951). (Russian)

The generalized Tricomi problem is considered for the equation $u_{xx}+\theta(y)u_{yy}=0$, $\theta(y)=1$ for $y>0$, $\theta(y)=-1$ for $y>0$. Let D be a domain bounded by an arc σ: $x=x(s)$,

$y=y(x)$ lying in the upper half-plane with end points at $(0, 0)$ and $(1, 0)$, by the line $y=x-1$ and by the curve L: $y=-\gamma(x)$, $0\leqq x\leqq l$, with $\gamma(0)=0$, $l+\gamma(l)=1$, and such that L lies inside the triangle formed by the points $(0, 0)$, $(1, 0)$, $(\frac{1}{2}, -\frac{1}{2})$. If boundary values are prescribed on σ and L, then there exists a unique solution in D provided that σ and L satisfy the restrictive conditions $y_s'(x-x^2-y^2)-x_s'y\geqq 0$, $|d\gamma/dx|<1$, $d\gamma/dx<\gamma(x-x^2+\gamma^2)^{-1}$. This continues the previous work of the author [Lavrent'ev and Bicadze, same Doklady (N.S.) **70**, 373–376 (1950); Bicadze, ibid. **70**, 561–564 (1950); these Rev. **11**, 724]. *M. H. Protter.*

Ladyženskaya, O. Solution of a mixed problem by means of finite differences. Doklady Akad. Nauk SSSR (N.S.) **85**, 705–708 (1952). (Russian)

For the hyperbolic equation

$$(*)\quad Lu+f=\sum_{i,j=0}^{n} a_{ij}(X, x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=0}^{n} a_i\frac{\partial u}{\partial x_i}+cu+f=0$$

the author treats the mixed initial-boundary-value problem in a cylinder in $(X\times x_0)$-space (x_0 is time). Cauchy data are given in some bounded domain in X-space and the value zero is prescribed on the wall of the cylinder. A class of testing functions is introduced; a generalized solution of (*) is a function satisfying the initial-boundary conditions and also satisfying an integral relation involving an arbitrary member of the class of testing functions. Existence and uniqueness theorems for generalized solutions are given with mild conditions on the coefficients and initial data. Estimates for the solution are also given. A finite-difference technique is employed. *M. H. Protter.* (Princeton, N. J.).

Bicadze, A. V. On the uniqueness of solution of a general boundary problem for an equation of mixed type. Soobščeniya Akad. Nauk Gruzin. SSR. **11**, 205–210 (1950). (Russian)

A boundary-value problem is treated for the equation (*) $\theta(y)U_{xx}+U_{yy}=0$ where $\theta(y)=1$ for $y>0$, -1 for $y<0$. Consider the domain D bounded by (i) the Jordan arc L: $x=x(s)$, $y=y(s)$ lying in the upper half-plane with endpoints $A(0, 0)$ and $B(1, 0)$. The arc L is assumed to be normal to the x-axis at its endpoints and to satisfy (everywhere) the condition $x_s'y+y_s'(1-x)<0$; (ii) the line L_1: $y=1-x$; (iii) the curve L_2: $y=-\gamma(x)$ such that $\gamma(0)=0$, $|\gamma'(x)|<1$, $\gamma'(x)\geqq-\gamma(1-x)$, and which intersects L_1. Let u be a solution of (*) which is continuous in D and such that u_x and u_y may approach infinity of order less than one as $(x, y)\to(1, 0)$. It is shown that if u vanishes on L and L_1, it vanishes throughout D. *M. H. Protter* (Princeton, N. J.).

Protter, M. H. A boundary value problem for an equation of mixed type. Trans. Amer. Math. Soc. **71**, 416–429 (1951).

This paper deals with the elliptico-hyperbolic equation (*) $K(y)u_{xx}+u_{yy}=0$, $yK(y)\geqq 0$ which includes as special cases the equations of Chaplygin and Tricomi. Let D be a domain in $y<0$ bounded by the segment $(-a, a)$ of the x-axis and the two characteristics of (*), γ_1 and γ_2, emerging from the points $(-a, 0)$ and $(a, 0)$, respectively. The boundary-value problem consists in finding a solution u of (*) in D such that $u(x, 0)=F(x)$, $u(x, y)=G(x)$ at every point (x, y) on γ_2. Under the hypotheses that $K(y)$ is continuous non-decreasing, $K(0)=0$, and that $F(x)$, $G(x)$ are of class C^1, and $F(b)=G(b)$, the author proves that the problem has a unique solution, and that this solution satisfies the inequality

$$|u(x, y)|\leqq\max(|F|, |G|)+4|y|\max(|F'|, |G'|)(\max|K|)^{1/2}.$$

The method of proof involves the approximation of $K(y)$ by a step-function and the explicit construction of the solution of the resulting approximating equation. This method was used in treating the Cauchy problem for (*) [cf. the reviewer's paper, NACA Tech. Note, no. 2058 (1950); these Rev. **12**, 61], but in the present case a much more delicate reasoning is needed. *L. Bers* (New York, N. Y.)

Cinquini-Cibrario, Maria. Sopra alcuni problemi preliminari. I, II. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 49–59, 71–78 (1950).

These notes contain certain auxiliary theorems used by the author in an earlier paper [Ann. Scuola Norm. Super. Pisa (3) **3**, 161–197 (1950); these Rev. **12**, 337]. These auxiliary theorems, which are too involved to be stated here, concern existence of first order systems of equations referred to characteristic parameters, with boundary conditions prescribed on 2 intersecting curves, one of which may have to be determined from the conditions. *F. John.*

Cinquini-Cibrario, Maria. Alcuni nuovi teoremi di esistenza per equazioni non lineari di ordine n di tipo iperbolico. Ann. Scuola Norm. Super. Pisa (3) **5**, 329–353 (1951).

This is a supplement to results given by the author in 2 previous papers [Ann. Mat. Pura Appl. (4) **26**, 95–117 (1947); Ann. Scuola Norm. Super. Pisa (3) **3**, 161–197 (1950); these Rev. **10**, 379; **12**, 337]. Those papers dealt with the problem of finding an integral surface passing through a given "characteristic strip" [for the definitions see the cited reviews] and satisfying an additional linear differential equation along another curve C intersecting the strip. The second paper was restricted to systems of quasi-linear equations, while in the first paper the boundary condition along C had to be a rather special type. In the present paper existence and uniqueness of an integral surface of a non-linear nth order equation in 2 independent variables is proved under the conditions that the surface contains a given characteristic strip, and along a curve C, whose projection on the (x, y)-plane is given, satisfies a given nth order linear differential equation. (The theorem holds under suitable regularity assumptions and inequalities satisfied by the data.) This is proved by reducing it to a corresponding theorem for the system of quasi-linear equations obtained by differentiation of the original non-linear differential equation, and applying the results obtained previously. A second theorem states that there exists an integral surface passing through a given characteristic strip and satisfying a given linear differential equation along a characteristic curve C, which intersects the strip. Here C is not prescribed, but only belongs to a characteristic root distinct from that associated with the strip. Additional theorems deal with the degenerate case of a quasi-linear equation. *F. John* (New York, N. Y.).

Eichler, Martin. Eine Modifikation der Riemannschen Integrationsmethode bei partiellen Differentialgleichungen vom hyperbolischen Typ. Math. Z. **53**, 1–10 (1950).

The author traces in detail the formal analogy between Riemann's integration method for partial differential equations of hyperbolic type and the more recent method of representing the solutions of a partial differential equation

of elliptic type by means of a "generating solution" and arbitrary analytic functions of a complex variable (S. Bergman's integral operator method). Generalizing the procedure used in the elliptic case, he develops an integration method for hyperbolic equations which in certain respects may be regarded as simpler than Riemann's method. *Z. Nehari.*

Günther, Paul. Zur Gültigkeit des Huygensschen Prinzips bei partiellen Differentialgleichungen vom normalen hyperbolischen Typus. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **100**, no. 2, 43 pp. (1952).

A partial differential equation of normal hyperbolic type can be written in the form

$$(*) \qquad g^{\alpha\beta}\nabla_\alpha\nabla_\beta u + A^\alpha\nabla_\alpha u + Cu = 0$$

where ∇_α denotes covariant differentiation in a Riemannian space of indefinite metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. Hadamard conjectured that Huygens' principle holds in the strict sense for such an equation if and only if it is reducible to the ordinary equation of wave motions by coordinate-transformations, by multiplication by a factor $\sigma(x^\alpha)$, and/or by a change of independent variable $u = \lambda(x^\alpha)u'$. When this is the case, the equation represents pure waves without diffusion.

Mathisson [Acta Math. 71, 249–282 (1939); these Rev. 1, 120] proved this conjecture for the case when the Riemannian space is of four dimensions and the tensor $g_{\alpha\beta}$ is constant. He used, not Hadamard's method involving the elementary solution, but the method of the parametrix he had developed in an earlier paper [Math. Ann. **107**, 400–419 (1932)]. The condition for pure waves is that the parametrix satisfy the equation adjoint to (*).

The present paper deals with the case of four dimensions when the $g_{\alpha\beta}$ are functions of the coordinates. It is impossible to give a brief account of the complicated tensor analysis. The final result is the derivation of four necessary conditions for pure waves; the first two are

$$C - \tfrac{1}{2}\nabla_\alpha A^\alpha - \tfrac{1}{4}A_\alpha A^\alpha = \tfrac{1}{8}R,$$
$$\Box A_\alpha - \nabla_\alpha\nabla_\rho A^\rho + R_{\alpha\rho}A^\rho = 0,$$

where $R_{\alpha\rho}$ is the contracted curvature tensor, R the scalar curvature; the other two involve the conformal curvature tensor. Whether these conditions do imply Hadamard's conjecture is not discussed. *E. T. Copson.*

Magenes, Enrico. Sull'equazione del calore: teoremi di unicità e teoremi di completezza connessi col metodo di integrazione di M. Picone. I. Rend. Sem. Mat. Univ. Padova **21**, 99–123 (1952).

Magenes, Enrico. Sull'equazione del calore: teoremi di unicità e teoremi di completezza connessi col metodo d'integrazione di M. Picone. II. Rend. Sem. Mat. Univ. Padova **21**, 136–170 (1952).

Let D be a domain in the (x, y)-plane bounded by $k+1$ closed curves $C_0, C_1, \cdots, C_k$ of class 2. The domain interior to the curve C_0 contains the non-intersecting curves $C_1, \cdots, C_k$. Denote by τ the region generated by moving D parallel to the z-axis from $z=0$ to $z=z_0>0$. Let $\sigma = s + p(0) + p(z_0)$ denote the boundary of τ, with $p(0)$ and $p(z_0)$ denoting those parts of σ that lie respectively in the planes $z=0$ and $z=z_0$. The letter n will be used to denote the interior normal to the surface σ. The fundamental solution of

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial z} = 0$$

with the normalizing factor of $1/4\pi$ will be written $G(x, y, z; \xi, \eta, \zeta) = G(P; M)$. If $A(M)$ and $B(M)$ are summable on σ, the following integral exists:

$$Q(P; A, B) = \int_\sigma \left\{ \left[A(M)\frac{\partial G(P, M)}{\partial n_M} - B(M)G(P, M) \right] \sin(n, \zeta) + A(M)G(P, M)\cos(n, \zeta) \right\} d_M\sigma.$$

A solution u of (1) in τ will be said to belong to class Γ if the following three conditions are satisfied:

$$\text{(a)} \quad \lim_{P\to M} u(P) = A(M), \qquad \text{(b)} \quad \lim_{P\to M} \frac{\partial u}{\partial n_M} = B(M),$$

the limits holding almost everywhere as P approaches the point M on σ along the interior normal at M, and (c) the summable functions A and B are such that $Q(P; A, B) = 0$ for P exterior to τ, $Q(P; A, B) = u(P)$ for P interior to τ. We shall refer to A and B as the limit functions associated with u.

In the first paper the following two uniqueness theorems are proved. If $u \in \Gamma$, and the associated limit function A of u is zero on $s + p(0)$, then $u \equiv 0$ in τ. If $u \in \Gamma$, then $u \equiv 0$ in τ if its associated limit functions A and B are such that $A = 0$ on $p(0)$ and $B(M) - h(M)A(M) = 0$ on s. Here h is some bounded function continuous almost everywhere on s. Using these results the author shows that certain sets of functions are complete in L^2 on $p(z_0) + s$. In particular, if the domain D generating τ is bounded only by curve C_0, then the following set of functions is complete in the class of square summable functions on $p(z_0) + s$:

$$(2) \qquad \left(\frac{\partial^{i+j} f}{\partial\alpha^i \partial\beta^j} \right)_{\alpha=\beta=0}, \quad i, j = 1, 2, \cdots,$$

where $f = \exp(\alpha x + \beta y - (\alpha^2 + \beta^2)z)$.

In the second paper similar results are obtained based on the following uniqueness result. Let $s = s^1 + s^{11}$ where s^1 and s^{11} are measurable subsets of s, and let $H(M)$ have the properties assigned above to $h(M)$ with the additional property that $H \geqq 0$ on s^{11}. If $u \in \Gamma$ and has A and B as its associated limit functions, then $u = 0$ on τ if $A = 0$ on $p(0) + s^1$ and $B(M) - H(M)A(M) = 0$ on s^{11}. Several theorems are stated relative to Stieltjes integrals of which the following is a typical example. Let the function $\alpha(e)$ be additive and of limited variation for Borel sets e on s. Let δ be a set on s with boundary $F\delta$, and indicate by δ_ρ a surface in τ parallel to the surface δ and at a distance ρ. If $\alpha(F\delta) = 0$, then

$$\lim_{\rho\to 0} 2\int_{\delta_\rho} d_P s \int_\sigma \frac{\partial G(P, M)}{\partial n_M} d_M\alpha = \alpha(\delta) + 2\int_\delta d_N s \int_\sigma \frac{\partial G(N, M)}{\partial n_M} d_M\alpha.$$

Completeness theorems are also obtained in the class of continuous functions. In particular, if D is simply connected, the set of functions (2) is shown to be complete in the class of functions continuous on $p(z_0) + s$. *F. G. Dressel.*

Ficken, F. A. Uniqueness theorems for certain parabolic problems. J. Rational Mech. Anal. **1**, 573–578 (1952).

Let the variables y_i have the range $-\infty < y_i < \infty$ and the z_j have the ranges $a_j \leqq z_j \leqq b_j$. If for the continuous function $f(z_1, \cdots, z_n; y_1, \cdots, y_k) = f(z, y)$ there exists a function $M(s_1, \cdots, s_k) = M(s)$ such that $|f(z, y)| \leqq M(s) < \infty$ when $|y_i| \leqq s_i < \infty$, we shall write $f \in B(a_j \leqq z_j \leqq b_j)$. Let A denote

the set of points (x, t) such that $0 \leqq x \leqq 1$, $0 < t$, and consider the nonlinear boundary value problem

$$u_{xx} - u_t = N(x, t, u, u_x), \quad (x, t) \in A,$$
$$(1) \quad u_x(\alpha, t) - (-1)^\alpha a_\alpha(t) u_t(\alpha, t) = n_\alpha(t, u(\alpha, t)), \quad 0 < t, \ \alpha = 0, 1,$$
$$u(x, 0+) = g(x), \quad 0 \leqq x \leqq 1.$$

Here $N(x, t; y, z)$, N_y, $N_z \in B(0 \leqq x \leqq 1, 0 < t)$, the $a_\alpha(t)$ are continuous for $t > 0$, and $n_\alpha(t, y)$, $n_{\alpha y} \in B(0 < t)$. Under these conditions the following uniqueness theorem is proved: If $a_\alpha(t) \geqq 0$ and $g(x)$ is continuous, then there exists at most one solution $u(x, t)$ of the system (1) such that u, u_x, u_{xx}, u_t are continuous and u, u_x are uniformly bounded on A. Removing the restrictions that u_x is uniformly bounded on A and varying somewhat the restrictions on N and the a_α, the author obtains a uniqueness theorem with the lighter requirement on the function $g(x)$ that it is bounded and of the first Baire category. *F. G. Dressel* (Durham, N. C.).

Landis, E. M. On uniqueness of solution of Cauchy's problem for a parabolic equation. Doklady Akad. Nauk SSSR (N.S.) **83**, 345–348 (1952). (Russian)

The following theorem is proved: If the real-valued function $u(x, y)$ is defined on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, is twice continuously differentiable, and satisfies in this square the partial differential equation $\partial^2 u/\partial x^2 = a(x, y)\partial u/\partial y$, and the Cauchy conditions $u(0, y) = 0$, $\partial u(0, y)/\partial x = 0$ for $0 \leqq y \leqq 1$, where the function $a(x, y)$ is bounded, does not change sign, and satisfies a Lipschitz condition with respect to y in the same square, then $u(x, y) \equiv 0$. *J. B. Diaz.*

Kamynin, L. I. On the convergence of a finite-difference process for the heat conduction equation. Doklady Akad. Nauk SSSR (N.S.) **85**, 701–703 (1952). (Russian)

In a preceding paper [Doklady Akad. Nauk SSSR (N.S.) **82**, 13–16 (1952); these Rev. **14**, 172] the author considered the difference between the uniqueness theorems for solutions of the heat equation

$$(*) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \varphi(x), \quad -\infty < x < \infty,$$

and the solutions of the corresponding infinite system of differential-difference equations

$$(**) \qquad \frac{\partial u^{(h)}(x, t)}{\partial t} = \frac{1}{h^2}[u^{(h)}(x+h, t) - 2u^{(h)}(x, t) + u^{(h)}(x-h, t)],$$
$$u^{(h)}(x, 0) = \varphi(x), \quad x = \cdots, -2h, -h, 0, h, 2h, \cdots.$$

According to a theorem of A. N. Tihonov [Mat. Sbornik **42**, 199–216 (1935)] a solution of (*) is determined uniquely in the class of functions satisfying the growth condition (T and C are positive constants)

$$\max_{0 \leqq t \leqq T} |u(x, t)| e^{-Cx^2} \to 0 \quad \text{as} \quad |x| \to \infty,$$

while the author showed in his preceding paper that the uniqueness of a solution of (**) holds for the class of functions satisfying

$$|u(x, t)| \leqq A[(1-\epsilon)|x|/h]!,$$
$$x = \cdots, -2h, -h, 0, h, 2h, \cdots; \ 0 \leqq t \leqq T; \ 0 < \epsilon \leqq 1,$$

and that uniqueness fails to hold if

$$|u(x, t)| \leqq A[(1+\epsilon)|x|/h]!$$

so that the domain of uniqueness of (**) is narrower than than of (*). In the present paper the author considers the convergence of solutions of (**) to those of (*). It is shown first that if $\varphi(x)$ is continuous on $-\infty < x < \infty$ and satisfies

$$(\dagger) \qquad |\varphi(x)| \leqq A e^{C|x| \ln (1+|x|)},$$

where A and C are positive constants, then the solution of (**) converges as $h \to 0$ to Poisson's integral

$$\frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\varphi(\xi)}{t^{1/2}} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi,$$

which is the corresponding solution of (*). In general, if $\varphi(x)$ satisfies Tihonov's condition

$$|\varphi(x)| \leqq A e^{C|x|^{2-\delta}}, \quad \delta > 0,$$

but not (†), there may not be convergence as $h \to 0$. However, the author proves two theorems which, in general terms, state that in this case, if $\varphi(x)$ is first truncated before passing to the limit [i.e., by considering functions which coincide with $\varphi(x)$ for $-X \leqq x \leqq X$ and vanish for $|x| > X$, and the corresponding solutions of (**)] and h is chosen sufficiently small in relation to the "truncation parameter" X, then the desired convergence as $h \to 0$ is obtained.

J. B. Diaz (College Park, Md.).

Thiruvenkatachar, V. R., and Ramakrishna, B. S. A case of combined radial and axial heat flow in composite cylinders. Quart. Appl. Math. **10**, 255–262 (1952).

Let $u_1(r, x, t)$ denote the temperatures at points in a cylinder $0 \leqq r \leqq a$, $0 \leqq x \leqq l$ whose material has constant thermal coefficients, and let $u_2(r, x, t)$ denote the temperatures in a coaxial shell $a \leqq r \leqq b$, $0 \leqq x \leqq l$, in perfect contact with the cylinder at the surface $r = a$. The constant thermal coefficients of the shell may differ from those of the core. The composite cylinder so formed has unit temperature throughout initially; its surface $r = b$ is thermally insulated and its bases $x = 0$ and $x = l$ are kept at temperature zero after the instant $t = 0$. With the aid of Laplace transforms and Fourier methods the authors derive formulas that represent the temperatures u_1 and u_2 in series involving Bessel functions and elementary functions. Results of numerical computation are presented graphically to illustrate how u_1 and u_2 vary with r at different times and positions for one set of values of the thermal coefficients such that the core has a conductivity ten times that of the shell. [Reviewer's remark: the graphical results shown in Fig. 3 seem to be of the type that might be anticipated if the conductivity of the shell were larger than that of the core.]

R. V. Churchill (Ann Arbor, Mich.).

Satoh, Tunezo. On the mathematical analysis of the problem of the heat conduction with the variable transfer coefficient. J. Phys. Soc. Japan **7**, 245–249 (1952).

Let $v(r, t)$ denote the temperatures inside a long cylinder $r < 1$ that is initially at unit temperature throughout. At the surface $r = 1$ heat transfer takes place into a medium at temperature zero according to a linear law $v_r = -hv$ where the coefficient of transfer h is a function of time t. The author shows that the temperature $H(t)$ of the surface of the cylinder is the solution of a certain linear integral equation involving $h(t)$ and that the function $v(r, t)$ satisfies a linear integrodifferential equation involving $H(t)$. When h is a constant, some known results are obtained from those equations. *R. V. Churchill* (Ann Arbor, Mich.).

Fridlender, V. R. Investigation of the Kovalevskaya-Goursat problem for a class of linear differential-operator equations. Doklady Akad. Nauk SSSR (N.S.) **76**, 363–365 (1951). (Russian)

As a generalization of the Cauchy-Kowalevsky problem in the sense of Salehov [same Doklady (N.S.) **59**, 857–859 (1948); Izvestiya Akad Nauk SSSR. Ser. Mat. **14**, 355–366 (1950); these Rev. **9**, 441; **12**, 338], the author states two theorems which answer the following question: to determine necessary and sufficient conditions (on the structure of the initial data φ_{ik} below) in order that the equation

$$\frac{\partial^p z}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_s^{\alpha_s}} = f(x_1, \cdots, x_s)L(z) + F(x_1, \cdots, x_s; y_1, \cdots, y_t),$$

$p=\alpha_1+\alpha_2+\cdots+\alpha_s$, have a unique solution $z(x, y)$ analytic in the (real or complex) variables $x_1, \cdots, x_s$ in the neighborhood of the point $x_1=x_2=\cdots=x_s=0$ and satisfying the initial conditions

$$\left.\frac{\partial^k z}{\partial x_i^k}\right|_{x_i=0} = \varphi_{ik}(x, y), \quad i=1, 2, \cdots, s; k=0, 1, \cdots, \alpha_i-1,$$

where the "compatibility conditions" of the initial data:

$$\left.\frac{\partial^q \varphi_{ik}}{\partial x_j^q}\right|_{x_j=0} = \left.\frac{\partial^k \varphi_{jq}}{\partial x_i^k}\right|_{x_i=0}; \; i, j=1, 2, \cdots, s; i\neq j;$$

$$k=0, 1, \cdots, \alpha_i-1; q=0, 1, \cdots, \alpha_j-1,$$

are supposed to be identically satisfied. The linear operator L is supposed independent of $x_1, x_2, \cdots, x_s$ and to be defined for all continuous functions in $y_1, y_2, \cdots, y_t$ defined on a certain closed domain. *J. B. Diaz.*

Kokareva, T. A. Some existence theorems for analytic solutions of integro-differential equations. Doklady Akad. Nauk SSSR (N.S.) **79**, 13–16 (1951). (Russian)

Consider the equation

$$\frac{\partial^p z(x_1, \cdots, x_m, y_1, \cdots, y_s)}{\partial x_1^{p_1}\cdots\partial x_m^{p_m}} = \sum_{\gamma_1=0}^{r_1}\cdots\sum_{\gamma_m=0}^{r_m} A_{\gamma_1,\cdots,\gamma_m}(x_1, \cdots, x_m, y_1, \cdots, y_s) \times \frac{\partial^{\gamma_1+\cdots+\gamma_m} D_{y_{n+1},\cdots,y_s} z(x_1, \cdots, x_m, y_1, \cdots, y_s)}{\partial x_1^{\gamma_1}\cdots\partial x_m^{\gamma_m}} + f(x_1, \cdots, x_m, y_1, \cdots, y_s),$$

where $0\leqq n\leqq s$, $p=\sum_{i=1}^m p_i$; $r_i<p_i$ $(i=1, \cdots, m)$; and $D_{y_{n+1},\cdots,y_s}$ is an integral differential operator which takes continuous functions in the variables $y_{n+1}, \cdots, y_s$ into functions continuous in all the variables indicated. Impose further the initial conditions $(i=1, 2, \cdots, m; k_i=0, 1, \cdots, p_i-1)$

$$\left.\frac{\partial^{k_i} z(x_1, \cdots, x_m, y_1, \cdots, y_s)}{\partial x_i^{k_i}}\right|_{x_i=0} = \varphi_i^{k_i}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m, y_1, \cdots, y_s),$$

supposing that the compatibility conditions

$$\left.\frac{\partial^{k_j}\varphi_j^{k_i}}{\partial x_j^{k_j}}\right|_{x_j=0} = \left.\frac{\partial^{k_i}\varphi_j^{k_j}}{\partial x_i^{k_i}}\right|_{x_i=0}, \quad i, j=1, 2, \cdots, m; i\neq j;$$

$$k_j=0, 1, \cdots, p_j-1; k_i=0, 1, \cdots, p_i-1,$$

are identically satisfied. Necessary and sufficient conditions for the solution (for equations of the type above) of the Cauchy-Kowalevsky problem in the sense of G. Salehov have been considered by Salehov [Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 226–228 (1947); Doklady Akad. Nauk SSSR (N.S.) **59**, 857–859 (1948); Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 355–366 (1950); these Rev. **10**, 44; **9**, 441; **12**, 338] and by V. R. Fridlender [see the preceding review]. In the present note the author announces simple sufficient conditions for a problem of this type for the equation above. In some particular cases the conditions obtained turn out to be necessary also. *J. B. Diaz.*

Gerasimenko (Kuznecova), L. V. Solution of the Cauchy-Kovalevskaya problem for certain partial differential equations in a domain of arbitrarily smooth functions. Doklady Akad. Nauk SSSR (N.S.) **86**, 11–14 (1952). (Russian)

Consider the partial differential equation

$$\frac{\partial^{2p} u}{\partial t^{2p}} + a_1\frac{\partial^{p+q} u}{\partial t^p \partial x^q} + a_2\frac{\partial^{2q} u}{\partial x^{2q}} = f(t, x), \tag{1}$$

where a_1 and a_2 are constants. The purpose of the present note is to study the Cauchy-Kowalewsky problem for this equation with a view towards answering the following two questions. (a) What conditions on the function $f(t, x)$ and on the initial data

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \varphi_k(x), \quad k=0, 1, 2, \cdots, 2p-1, \tag{2}$$

are necessary and sufficient in order that there be a solution of the initial value problem (1), (2) which is analytic in t near $t=0$? (b) If such a solution exists, what is its nature with respect to the other variable, x? A similar problem for a certain class of equations was considered by G. S. Salehov [Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 226–228 (1947); these Rev. **10**, 44] and later by Fridlender [in the paper reviewed second above] and Kokareva [in the paper reviewed above]. *J. B. Diaz* (College Park, Md.).

Salehov, G. On Cauchy's problem for a class of partial differential equations in a region of arbitrarily smooth functions. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **1**, 63–74 (1948). (Russian)

For the partial differential equation

$$\partial^p z/\partial t^p - (\pm)t^m \partial^q z/\partial x^q = 0, \quad m\geqq 0,$$

with the initial conditions $\partial^k z/\partial t^k = \phi_k(x)$, $k=0, 1, 2, \cdots, p-1$, the author considers the problem of relating the analytic behavior of the solution to the smoothness, i.e., differentiability, properties of the initial functions. He utilizes a classification of infinitely differentiable functions due to Gevrey [Ann. Sci. École Norm. Sup. (3) **35**, 129–190 (1918)] and shows that a necessary and sufficient condition that the solution of the above equation be unique and analytic in t for some neighborhood of $t=0$ is that $\alpha\leqq p/q$, where α is the Gevrey index of $\phi_k(x)$. If $\alpha<p/q$, the solutions are entire. The proof depends upon a detailed investigation of the asymptotic behavior of various special functions. *R. Bellman* (Santa Monica, Calif.).

Salehov, G. S. On the asymptotic behavior of the integrals of certain linear partial differential equations near to the boundary of the region of their existence. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 77–86 (1950). (Russian)

In this paper the author continues his investigation in the paper reviewed above of the relation between the structural

properties of the solutions and initial functions of the above partial differential equations and establishes the converse of the theorem stated above. *R. Bellman.*

Lyance, V. È. On Cauchy's problem in a domain of functions of real variables. Ukrain. Mat. Žurnal 1, no. 4, 42–63 (1949). (Russian)

In the present paper the author obtains certain generalizations of the results of Petrovskiĭ [Bull. Univ. d'Etat Moscou. Sér. Internat. Sect. A. 1, no. 7 (1938)], concerning Cauchy's problem for the system

$$(1)\quad \frac{\partial u_i}{\partial t}-\sum_{j=1}^{N}\sum_{(k)}^{K}A_{ij}^{(k_1,\cdots,k_n)}(t)\frac{\partial^{k_1+\cdots+k_n}u_j}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}=f_i(t,x_1,\cdots,x_n),$$

$i=1,\cdots,N$, subject to the initial conditions

$$(2)\quad u_i|_{t=t_0}=\varphi_i(x_1,\cdots,x_n),\quad i=1,\cdots,N,$$

where $0\leqq t_0<T$, the solution being sought in the domain $t_0\leqq t<T$, $|x_i|<\infty$, while the coefficients $A_{ij}^{(k_1,\cdots,k_n)}(t)$ are continuous for $0\leqq t\leqq T$. Petrovskiĭ showed that if a certain so-called condition A holds, then uniqueness and existence theorems hold for the system (1), (2), under certain boundedness restrictions on $u_1,\cdots,u_N$. In the present paper the author shows that if condition A holds, then the existence theorem holds for functions $u_1,\cdots,u_N$, which together with their derivatives, grow no faster than $\exp(\sum|x_k|)^r$, where the number r $(0<r<1)$ depends on the coefficients of the system (1). A class of functions for which the uniqueness theorem holds is also characterized. A new integral representation of the solution of Cauchy's problem is also given. As in Petrovskiĭ's paper, the Fourier integral method is employed. *J. B. Diaz* (College Park, Md.).

Schmidt, Adam. Lineare partielle Differentialgleichungen mit konstanten Koeffizienten. J. Reine Angew. Math. 189, 160–167 (1951).

E. Lammel, in an unpublished lecture, using a generalization of the relationship between analytic function theory and plane potential theory, showed that the partial differential equation with constant coefficients

$$(1)\quad a_0\frac{\partial^n u}{\partial x^n}+a_1\frac{\partial^n u}{\partial x^{n-1}\partial y}+\cdots+a_n\frac{\partial^n u}{\partial y^n}=0,$$

has the general solution

$$(2)\quad u=\sum_{j=1}^{r}\{\varphi_{j1}(x+\lambda_j y)+y\varphi_{j2}(x+\lambda_j y)+\cdots+y^{m_j-1}\varphi_{jm_j}(x+\lambda_j y)\},$$

where the λ_j are the roots of the equation

$$(3)\quad a_0+a_1\lambda+\cdots+a_n\lambda^n=0,$$

and m_j their multiplicities. Lammel required both u and the otherwise arbitrary functions φ_{jk} to be analytic. The present paper contains a direct proof of the fact that (2) is the general solution of (1), it being assumed only that all partial derivatives of the nth order of u are continuous; the functions φ_{jk} corresponding to non-real roots of (3) turn out to be analytic, while those corresponding to real roots of (3) turn out to be n times continuously partially differentiable. An extension of the theorem to systems with constant coefficients $\partial v_k/\partial t+\sum_{l=1}^{n}a_{kl}\partial v_l/\partial x=0$ is also given.

J. B. Diaz (College Park, Md.).

Difference Equations, Special Functional Equations

Lawden, D. F. On the solution of linear difference equations. Math. Gaz. 36, 193–196 (1952).

Myškis, A. D. Supplementary bibliographical materials to the article "General theory of differential equations with a retarded argument." Translated by J. M. Danskin, Jr. Air Force Project Rand, Santa Monica, Calif., Rep. T-23, 11 pp. (1952).

Translated from Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 148–154 (1950); these Rev. 12, 416.

Mirolyubov, A. A. Solution of differential-difference equations with linear coefficients. Doklady Akad. Nauk SSSR (N.S.) 85, 1209–1210 (1952). (Russian)

The author states without proof a method for finding, by contour integration, a particular solution of the differential-difference equation $M[f(x)]=F(x)$, where

$$M[f(x)]=\sum_{p=0}^{n}\sum_{q=0}^{m}(a_{pq}x+b_{pq})f^{(p)}(x+h_q).$$

x here is complex along with the a_{pq}'s and b_{pq}'s; $a_{nm}\neq 0$ and $a_{n0}\neq 0$; $0=h_0<h_1<\cdots<h_m$; $F(x)$ analytic in some region. Turning to the homogeneous equation $M[f(x)]=0$, he shows that any analytic solution $f(x)$ holomorphic in the strip $q_2<\mathrm{Im}\,x<q_1$ may be written as the sum of two solutions analytic in the half-planes $\mathrm{Im}\,x<q_1$ and $\mathrm{Im}\,x>q_2$ respectively, the half-planes being slit in a certain way. He states growth and series representation theorems for $f_1(x)$ and $f_2(x)$. He makes use of (unavailable) results of Leont'ev [Trudy Gor'kovsk. Gos. Ped. Inst. Fiz.-Mat. Fak. 14, 3 (1951)]. *J. M. Danskin* (Santa Monica, Calif.).

Popken, J. An arithmetical theorem concerning linear differential-difference equations. Acad. Serbe Sci. Publ. Inst. Math. 4, 121–128 (1952).

The author uses the Lindemann-Weierstrass theorem and a result of Schürer [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 70, 185–240 (1919), Satz VI] to prove that if all the constants $A_{\mu\nu}$, ω_ν in the linear differential-difference equation

$$(1)\quad \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}A_{\mu\nu}y^{(\mu)}(1+\omega_\nu)=0$$

are algebraic, not all the $A_{\mu\nu}$'s are zero, the ω_ν are all different, and if the $n+1$ equations $\sum_{\mu=0}^{m}A_{\mu\nu}t^\mu=0$ $(\nu=0,1,\cdots,n)$ have only $t=0$ as a common root, then no entire transcendental function of exponential type $y(x)=\sum_{k=0}^{\infty}(c_k/k!)x^k$ with $c_k=O(q^k)$, $q>0$, satisfying (1), can have its coefficients $c_0,c_1,\cdots$ algebraic. *J. M. Danskin.*

Łojasiewicz, S. Solution générale de l'équation fonctionelle $f(f(\cdots f(x)\cdots))=g(x)$. Ann. Soc. Polon. Math. 24 (1951), 88–91 (1952).

Given a set E of arbitrary nature, a function $f(x)$ is said to have property p_E if $f(x)$ maps E onto itself in one-to-one manner. One can then define iterates of f of all integer orders: $f^n(x)$, $n=0,\pm1,\pm2,\cdots$. The author obtains the general solution of (1) $f^N(x)=g(x)$ $(x\in E)$, where N is a fixed positive integer and $g(x)$ has property p_E. The particular case (2) $f^N(x)=x$ is first studied. Let

$$1=n_0<n_1<\cdots<n_r=N$$

be the divisors of N. Decompose E into a union of disjoint

sets

$$(3)\quad E=\{M_1^0\}+\{M_1^1+\cdots+M^1_{n_1}\}+\cdots+\{M_1^r+\cdots+M^r_{n_r}\}$$

in such a way that for each $i=1,\cdots,r$ the sets $M_1^i,\cdots,M^i_{n_i}$ are of the same power. Let $\varphi_j^i(x)$ be a function that maps M_j^i one-to-one on M^i_{j+1} for $j=1,\cdots,n_i-1$ and $i=1,\cdots,r$, and let $\psi_j^i(x)$ be the function inverse to $\varphi_j^i(x)$. Then a solution of (2) is given by

$$f(x)=x,\ x\in M_1^0;\quad f(x)=\varphi_j^i(x),\ x\in M_j^i,\ j=1,\cdots,n_i-1;\quad i=1,\cdots,r;\quad f(x)=\psi_1^i(\cdots\psi^i_{n_i-1}(x)\cdots),\ x\in M^i_{n_i},\quad i=1,\cdots,r.$$

Conversely, every solution of (2) can be obtained in this way. Using the results for equation (2), it is then shown how to obtain the general solution of (1) by means of suitable decompositions analogous to, but more complicated than, those of (3). *I. M. Sheffer* (State College, Pa.).

Kuwagaki, Akira. Sur quelques équations fonctionnelles et leurs solutions caractéristiques. II. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 47–53 (1952).

The author treats the functional equation

$$(1)\quad f(x,y)=\lambda\sum_{i=1}^{m}\sum_{j=1}^{n}f\left\{\frac{1}{m}[x+(m-i)a+(i-1)b],\ \frac{1}{n}[y+(n-j)c+(j-1)d]\right\},$$

where m, n are given positive integers, λ is a real parameter, and $f(x,y)$ is the unknown function. The method (one of iteration) and the results parallel those of part I [same Mem. 26, 271–277 (1951); these Rev. 14, 53]. The most interesting case, where $\lambda=m^{p-1}\cdot n^{q-1}$ (p, q integers), is examined in detail. *I. M. Sheffer* (State College, Pa.).

Read, A. H. The solution of a functional equation. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 336–345 (1952).

The equation treated is (1) $f[z,\phi(g(z))]=\phi(z)$. Let ζ be a double point of $g(z)$ (i.e., $g(\zeta)=\zeta$) and such that g is regular in a neighborhood of $z=\zeta$ and $|g'(\zeta)|<1$. Suppose also that $f(z,w)$ is regular in a neighborhood of $(z,w)=(\zeta,\beta)$ where β is such that $f(\zeta,\beta)=\beta$, and that $|\partial f(\zeta,\beta)/\partial w|<|g'(\zeta)|^{-1}$. It is then shown that equation (1) has a unique solution $\phi(z)$ that is regular in a neighborhood of $z=\zeta$, with $\phi(\zeta)=\beta$. Moreover, this solution is the limit of a sequence of functions $\{\phi_n(z)\}$ defined by the iteration process

$$\phi_{n+1}(z)=f[z,\phi_n(g(z))],$$

where $\phi_1(z)$ is an arbitrary function regular at $z=\zeta$, with $\phi_1(\zeta)=\beta$. The proof utilizes the existence (which is established) of a solution of the simpler equation

$$f[z,\phi(z)]=\phi(sz)$$

(s=constant). *I. M. Sheffer* (State College, Pa.).

Aczél, János. Functional equations in applied mathematics. Magyar Tud. Akad. Mat. Fiz. Oszt. Közlemények 1, 131–142 (1951). (Hungarian)

The author discusses various applications of functional equations in the theory of probability and mechanics.

P. Erdös (Los Angeles, Calif.).

Functional Analysis, Ergodic Theory

***Riesz, Frédéric, et Sz.-Nagy, Béla. Leçons d'analyse fonctionnelle.** Académie des Sciences de Hongrie, Akadémiai Kiadó, Budapest, 1952. viii+449 pp. $7.70.

This book is divided into two parts, modern theories of differentiation and integration, and spectral theory of linear transformations in Hilbert space; a long chapter on integral equations serves as a bridge between these two topics.

Chapter I defines zero measure, and works from the Lebesgue theorem that every monotone function has a derivative almost everywhere, to a study of interval functions, used to discuss Riemann integrals and bounded variation.

Chapter II defines the Lebesgue integral in terms of limits of monotone sequences of step functions; the whole presentation is made to depend on two lemmas in such a way that the generalization to Lebesgue-Stieltjes integrals and to n dimensions can be made by a simple transition principle. Measurable functions and measure are defined; absolute continuity of the integral is proved along with the Lebesgue decomposition of a function of bounded variation. L^p spaces, mean and weak convergence, and the Riesz-Fischer theorem are followed by a comparison of this approach with the original definition by way of measure.

Chapter III discusses Stieltjes integrals and the representation of linear functionals on the space C of continuous functions, describes the Lebesgue-Stieltjes integral in n dimensions and the Daniell integral, and gives one variant of the Radon-Nikodým theorem.

Chapter IV treats integral equations by two methods, with an application to potential theory. Chapter V defines Hilbert and Banach spaces and gives a condition for complete continuity of a linear transformation of C into itself. Chapter VI studies symmetric, completely continuous transformations in Hilbert space, with application to the vibrating string and to almost periodic functions.

Chapter VII begins with two proofs of the spectral decomposition of a bounded symmetric operator in Hilbert space, then extends these results to unitary and normal transformations. Chapter VIII gives the corresponding decomposition for unbounded, self-adjoint transformations and discusses extensions of symmetric transformations. Chapter IX describes a functional calculus for self-adjoint transformations and discusses the behavior of the spectrum under perturbation.

Chapter X gives Stone's theorem on groups of unitary operators, discusses groups and semigroups of operators, and concludes with proofs of mean ergodic theorems in Hilbert space. Chapter XI defines a functional calculus for not-necessarily-normal transformations by means of line integrals in the resolvent set, and closes with a discussion of spectral sets of such a transformation.

The book ends with a ten-page bibliography, an index, a list of notations, and a page of errata. (The authors wish to mention another error; on p. 33, line 6 from below the word "soit" should be replaced by "=presque partout une fonction".)

This book shows the authors' skill in choosing for each result the proof best suited to give a student sufficient understanding to make further applications and extensions appear easy and natural. Careful references are made throughout, both for the material in the text and for related topics. This book covers so much material so well that it should be noted that it does not discuss rings of operators in Hilbert space nor spectral multiplicity. But a student interested in inte-

gration or spectral theory will find in this book much pertinent background material as well as a clear, concise discussion of these major topics. *M. M. Day.*

Carafa, Mario. Sulle regioni connesse dello spazio funzionale analitico e sulla rappresentazione dei funzionali polinomiali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 478–492 (1950).

Fantappiè a défini dans son espace fonctionnel $\mathfrak{S}^{(1)}$ un concept de "ensemble connexe" un peu divers de celui généralement adopté: un ouvert $\mathfrak{A}$ di $\mathfrak{S}^{(1)}$ est dit connexe, si, étant donnés $y_1, y_2 \in \mathfrak{A}$, il existe dans $\mathfrak{A}$ une ligne continue d'extrêmes y_1, y_2, constituée par des arcs de ligne analytique (en nombre fini). Soit $\mathfrak{H}$ un ouvert connexe de $\mathfrak{S}^{(1)}$, dont chaque point admet un voisinage connexe contenu dans $\mathfrak{H}$. L'auteur démontre que, pour chaque couple d'éléments y_1, y_2 de $\mathfrak{H}$, il existe une chaîne finie de voisinages connexes contenus dans $\mathfrak{H}$, tels que le premier contient y_1 et le dernier contient y_2. En outre, il démontre que, si F est une fonctionnelle analytique définie dans $\mathfrak{H}$, les valeurs de F dans $\mathfrak{H}$ sont déterminées par les valeurs de F dans un voisinage quelconque contenu dans $\mathfrak{H}$. Enfin, soit $\mathfrak{H}_1$ un ouvert connexe de $\mathfrak{S}^{(1)}$, dont chaque point admet un voisinage linéaire connexe $(A) \subset \mathfrak{H}_1$ et soit F_1 une fonctionnelle analytique polinomiale définie dans $\mathfrak{H}_1$; alors, si l'on identifie chaque fonction holomorphe de $\alpha_1, \cdots, \alpha_k$ avec tous ses prolongements analytiques, on peut affirmer que F_1 est représentable dans $\mathfrak{H}$ par un développement "unique" de Fantappiè,

$$F_1[y(t)] = F_1[O_A] + \sum_{k=1}^{\infty} \frac{1}{k!} F_1^{(k)}[O_A; \overset{*}{\alpha}_1, \cdots, \overset{*}{\alpha}_k] y(\underset{*}{\alpha_1}) \cdots y(\underset{*}{\alpha_k}),$$

où O_A désigne la fonction identiquement nulle de domaine A.

J. Sebastião e Silva (Lisbonne).

Van Hove, Léon. Topologie des espaces fonctionnels analytiques et des groupes infinis de transformations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 333–351 (1952).

Some topological aspects of the spaces A_K of functions of complex variables analytic in a neighborhood of a given compact set K were already studied by several authors, mostly with emphasis on countable limits or unpleasant topological structures. Surprisingly enough, only recently, after the appearance of a paper by Dieudonné and Schwartz [Ann. Inst. Fourier Grenoble **1**, 61–101 (1950); these Rev. **12**, 417], has it become clear how the spaces A_K could be included in a simple and satisfactory way in the usual theory of topological vector spaces. In the present paper, the author's aim is to indicate how this can be done and to derive a few topological properties of A_K and of the group of analytic and analytically invertible mappings of K onto itself. As the author points out, other authors (namely Grothendieck, Köthe, and Silva Dias) have also simultaneously and independently applied the ideas of Dieudonné and Schwartz to this special case. *L. Nachbin.*

Edwards, R. E. The translations of a function holomorphic in a half-plane. II. Proc. London Math. Soc. (3) **2**, 279–285 (1952).

This paper supplements a previous one by the same author [same Proc. (3) **1**, 118–128 (1951); these Rev. **13**, 44]. E is the class of functions $f(\sigma+i\tau)$ defined and holomorphic when $\sigma>0$ and such that, for each $a>0$,

$$\sup_{\sigma>a} \int_{-\infty}^{\infty} |f(\sigma+i\tau)|^2 d\tau < \infty.$$

E is given a certain convex topology, and the class E' of continuous linear functionals on E is identifiable with the class of functions $u(x)$ defined a.e. on $(0, \infty)$ and such that $e^{\alpha x} u(x) \in L^2(0, \infty)$ for some $\alpha>0$, the α depending on u.

The first main result of the present paper is that, with the weak topology of E defined in the usual way in terms of E', every bounded and weakly closed set in E is weakly compact. Next the author considers the finest locally convex topology on E' relative to which E is exactly the topological dual of E'. This is called the κ topology by Arens [Duke Math. J. **14**, 787–794 (1947); these Rev. **9**, 241; **10**, 855] and the $\tau(E', E)$ topology by Schwartz and Dieudonné [Ann. Inst. Fourier Grenoble **1**, 61–101 (1950); these Rev. **12**, 417]. Owing to the foregoing result on weak compactness in E, the κ-topology is the same as the b-topology of Arens and the $\beta(E', E)$ topology of Schwartz and Dieudonné. This latter topology is called the 'strong' topology of E'. As a result, the author strengthens Theorem III of his original paper by using the strong instead of the weak topology of E' in the theorem. The theorem is a characterization of the closed linear subspace of E' spanned by elements of the form $e^{-\alpha x} u(x)$, where u is a fixed member of E', and α varies over a certain kind of point set. Another theorem of the original paper is supplemented by drawing upon the theory of quasi-analytic classes of functions.

A. E. Taylor (Los Angeles, Calif.).

Nevanlinna, Rolf. Über metrische lineare Räume. I. Allgemeine Bemerkungen zur Metrisierbarkeit. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **108**, 8 pp. (1952).

This is the first in a series of papers [see the following review]. Introducing the series, the author writes "Obwohl man verschiedene Wege kennt die lineare oder affine Geometrie axiomatisch aufzubauen, scheint das Problem der Einführung einer Metrik nicht restlos aufgeklärt zu sein, nicht einmal für den einfachsten Fall der endlichdimensionalen affinen Geometrie. Ein näheres Studium dieser Frage führt zu gewissen Ergebnissen, die im Folgenden mitgeteilt werden sollen."

This first paper is introductory and expository. After describing various axiomatic approaches to affine geometry, the author considers the addition of a congruence relation $\cong$ for segments, requiring as a minimum that $\cong$ be an equivalence and be preserved under every "Parallelverschiebung." He then considers (in two dimensions) properties of $\cong$ implied by further restrictions, such as the existence of three vectors which are pairwise linearly independent and congruent. *V. L. Klee, Jr.*

Nevanlinna, Rolf. Über metrische lineare Räume. II. Bilinearformen und Stetigkeit. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **113**, 9 pp. (1952).

(This work is related to two papers by Menger [Revista Ci., Lima **50**, 155–165 (1948); Canadian J. Math. **1**, 94–104 (1949); these Rev. **10**, 549, 306].) Let R be a complex vector space and $Q(x, y)$, $(x, y) \in R \times R$, a self-adjoint bilinear form. The author seeks a general method of topologizing R by use of Q, requiring that the topology be translation-invariant and that Q have certain familiar continuity properties when R is finite-dimensional and also when (R, Q) is a Hilbert space (i.e., when the associated quadratic form $Q(x, x)$, $x \in R$, is positive-definite). He observes that $x_n \to 0$ must then imply [$Q(x_n, x_n) \to 0$ and $Q(x_n, y) \to 0$ for each $y \in R$], but rejects this as a definition of convergence because it does not insure continuity of all forms of the important type

$Q(x, y)=\sum_1^n \epsilon_i x_i y_i$, $(x, y) \in P \times P$, where $\epsilon_i = \pm 1$. He promises that in a subsequent note, by an analysis of the orthogonalization problem, he will arrive at a definition of the topology such that all forms of the above type are continuous.

V. L. Klee, Jr. (Charlottesville, Va.).

Lorch, E. R. Convexity and normed spaces. Acad. Serbe Sci. Publ. Inst. Math. **4**, 109–112 (1952).

Suppose $\mathfrak{M}$ is a real vector space with real-valued norm $\| \ \|$ satisfying: (1) $\|f\| \geqq 0$, $\|f\|=0$ if and only if $f=0$; (2) if $\alpha>0$, $\|\alpha f\|=\alpha\|f\|$; (3) $\|f+g\| \leqq \|f\|+\|g\|$; (4) there is a constant $c>0$ such that for all $f \in \mathfrak{M}$, $c^{-1}\|f\| \leqq \|-f\| \leqq c\|f\|$; (5) $\mathfrak{M}$ is complete in the metric $\|f-g\|+\|g-f\|$. Then $\mathfrak{M}$ is called a "space of type M". (For $c=1$, $\mathfrak{M}$ is a Banach space.) The author shows that the Hahn-Banach theorem is valid in $\mathfrak{M}$, that the conjugate space $\mathfrak{M}^*$ is of type M, and that the natural map of $\mathfrak{M}$ into $\mathfrak{M}^{**}$ is an algebraic and metric isomorphism. Applying these ideas, he gives a proof similar to Mazur's of the support property of convex bodies and obtains also a result on convex hulls. [Reviewer's note: The author asserts (Theorem 2) that if $\| \ \|$ satisfies (1) and (4) and the Hahn-Banach theorem holds, then $\| \ \|$ also satisfies (2) and (3). With his definition of the norm of a linear functional, this is erroneous, a counter-example being the real line with $\|x\|=|x|^{1/2}$. The error comes in the sentence "by the usual reasoning involving the H-B theorem," for positive homogeneity of $\| \ \|$ is here needed to obtain non-trivial bounded linear functionals on the one-dimensional subspaces of $\mathfrak{M}$. However, if $\|F\|$ (for a linear functional F on $\mathfrak{M}$) is defined as $\sup\{F(f) \mid \|f\|=1\}$ rather than $\sup\{F(f)/\|f\| \mid f \neq 0\}$, Theorem 2 is correct.]

V. L. Klee, Jr. (Charlottesville, Va.).

Gál, I. S. Sur la méthode de résonance et sur un théorème concernant les espaces de type (B). Ann. Inst. Fourier Grenoble **3** (1951), 23–30 (1952).

Let $u_n(x)$ be a sequence of homogeneous bounded opertors from a Banach space E to a linear normed space with $|u_n|=\sup_{\|x\| \leq 1}\|u_n(x)\|<+\infty$. Then $\limsup \|u_n(x)\|<+\infty$ on E implies (by the usual category argument) that the $|u_n|$ are bounded if the u_n are subadditive, that is, if $\|u_n(x+y)\| \leqq \|u_n(x)\|+\|u_n(y)\|$. It is shown here (by Lebesgue's "method of the gliding hump") that this is still true if the u_n are "asymptotically subadditive", that is, if $\|u_n(x+y)\| \leqq \|u_n(x)\|+O(|u_n| \cdot \|y_n\|)$ uniformly for $x, y \in E$ and $\inf_{\|y\| \leq 1}\{\|u_n(x+y)\|+\|u_n(x)\|-\|u_n(y)\|\} \geqq o(|u_n|)$ for each $x \in E$.

G. G. Lorentz (Toronto, Ont.).

Wolf, František. Analytic perturbation of operators in Banach spaces. Math. Ann. **124**, 317–333 (1952).

This is a continuation of the papers on perturbation theory by F. Rellich and the reviewer, especially of the reviewer's paper, Comment. Math. Helv. **19**, 347–366 (1947) [these Rev. **8**, 589]. It considers the analytic perturbation of linear transformations of a Banach space $\mathfrak{B}$ in itself and it was written without acquaintance with the reviewer's recent paper on the same subject [Acta Sci. Math. Szeged **14**, 125–137 (1951); these Rev. **13**, 849]. The main tool is, as in the cited paper, the contour integration of the resolvent. The main result reads as follows: Let

$$A(\epsilon)=A_0+\epsilon A_1+\cdots+\epsilon^n A_n+\cdots$$

be a bounded linear transformation of $\mathfrak{B}$ depending on the complex parameter ϵ in a neighborhood of $\epsilon=0$ (an "analytic function" of ϵ). Let $E(\epsilon)$ be an idempotent bounded linear transformation of ϵ, depending also analytically on ϵ and such that $A(\epsilon)E(\epsilon)=E(\epsilon)A(\epsilon)$. Denoting by $\mathfrak{M}_0$ the range of $E(\epsilon)$, suppose that, in $\mathfrak{M}_0$, the coefficients $A_0, \cdots, A_{N-1}$ are all scalar multiples of the identity, i.e., for $f \in \mathfrak{M}_0$, $A_k f=\mu_k f$ $(k=0, \cdots, N-1)$, and suppose that the spectrum of $E(0)A_N$, when considered as a transformation of $\mathfrak{M}_0$ in itself, splits off into the mutually disconnected sets $\sigma_1, \cdots, \sigma_m$. Then there exist idempotent linear transformations $F_k(\epsilon)$ $(k=1, \cdots, m)$, analytic in a neighborhood of $\epsilon=0$, and such that (a) the subspaces $\mathfrak{N}_k(\epsilon)=F_k(\epsilon)\mathfrak{B}$ reduce $A(\epsilon)$, (b) for every sufficiently small open set S containing σ_k there is a $\delta>0$ such that, for $|\epsilon|<\delta$, a point μ of the spectrum of $A(\epsilon)$ belongs to the spectrum of $A(\epsilon)$ when considered in $\mathfrak{N}_k$ if and only if

$$\left(\mu-\sum_{l=0}^{N-1}\mu_l\epsilon^l\right)\epsilon^{-N} \in S.$$

From this result, Rellich's theorem on the analytic perturbation of self-adjoint transformations may easily be derived, at least if the transformations are bounded.

B. Sz.-Nagy (Szeged).

Glicksberg, Irving. The representation of functionals by integrals. Duke Math. J. **19**, 253–261 (1952).

Let $C(X)$ denote the linear space of bounded continuous real-valued functions over a topological space X. The author considers non-negative linear functionals J on $C(X)$. The main results are, essentially, as follows: I) A given J has the form (1) $J(f)=\int f(x)m(dx)$, m being a measure on X, if and only if $J(f_n) \to J(f)$ whenever $f_n \in C(X)$, $f \in C(X)$, $f_n \leqq f_{n+1}$, $f_n(x) \to f(x)$ for every $x \in X$. II) If X is completely regular, then in order that all J have the form (1) it is necessary and sufficient that X be pseudo-compact [cf. Hewitt, Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126]. The first of these results is a special case of a result by Goldstine [Bull. Amer. Math. Soc. **47**, 615–620 (1941), see Corollary 1 on p. 619; these Rev. **3**, 207].

M. Katětov (Prague).

Nachbin, Leopoldo. Linear continuous functionals positive on the increasing continuous functions. Summa Brasil. Math. **2**, 135–150 (1951).

Let ν be a positive measure on the triangle

$$T=\{(x, y): 0 \leqq x<y \leqq 1\}.$$

Then the (not necessarily positive) measure μ determined on the segment $0 \leqq t \leqq 1$ by the relation (valid for all continuous $f(t)$)

$$\int_0^1 f(t)d\mu(t)=\int\int_T [f(y)-f(x)]d\nu(x, y)$$

possesses the property that every increasing continuous real-valued function $f(t)$ has a positive integral with respect to μ; moreover $\|\nu\| \geqq \frac{1}{2}\|\mu\|$ (the norm meaning the total variation). The author shows that, conversely, given a measure μ on $[0, 1]$ with the above property, there exists a positive measure ν on T which determines μ and such that $\|\nu\|=\frac{1}{2}\|\mu\|$. It is shown, moreover, that this theorem remains valid also if one replaces the segment $[0, 1]$ by an arbitrary compact ordered space E, i.e., a compact space with an order relation such that the "diagonal" $x=y$ is a closed subset of the topological space $E \times E$ of the ordered pairs (x, y) of elements of E; the rôle of T is then played by the set of those pairs (x, y) for which $x<y$. Among the lemmas and theorems which lead to this general theorem there are many interesting in themselves, such as, e.g., the following:

Let φ be an upper semi-continuous and ψ a lower semi-continuous real-valued function on the compact ordered space E. Assume that $\varphi \leqq \psi$ and that φ or ψ is increasing. Then there is an increasing continuous real-valued function f on E such that $\varphi \leqq f \leqq \psi$. *B. Sz.-Nagy* (Szeged).

Weston, J. D. On the bounds of a bilinear form related to Hilbert's. Quart. J. Math., Oxford Ser. (2) **3**, 115–118 (1952).

Let H be the Hilbert space whose vectors x are doubly infinite sequences of complex numbers x_n such that $\sum_{-\infty}^{\infty} |x_n|^2 < \infty$, with the inner product $\sum_{-\infty}^{\infty} x_n \bar{y}_n$. If λ is real and not an integer, denote by U_λ the linear transformation of H with the matrix

$$U_{\lambda;m,n} = \frac{\sin \pi\lambda}{\pi(m-n+\lambda)}.$$

It is easy to see that U_λ is a unitary transformation; thus $|(x, U_\lambda y)| \leqq \|x\| \cdot \|y\|$, equality occurring if and only if $U_\lambda y$ is proportional to x. If, in particular, $x_n = y_n = 0$ for $n \leqq 0$, this yields

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m \bar{y}_n}{m-n+\lambda} \right| \leqq \pi |\operatorname{cosec} \pi\lambda| \cdot \|x\| \cdot \|y\|$$

(extension of Schur's result [J. Reine Angew. Math. **140**, 1–28 (1911)] to complex vectors). Moreover, it follows easily that the equality sign never holds if all the x_n, y_n are non-negative real numbers, but the constant is the best even in this case. Let now V_λ be the transformation with the matrix

$$V_{\lambda;m,n} = \frac{\sin(m-n+\lambda)}{\pi(m-n+\lambda)} = (-1)^{m+n} U_{\lambda;m,n};$$

it is also unitary. Then, putting $x(t) = \sum_{-\infty}^{\infty} x_n e^{int}$ and using Parseval's theorem, one gets for all $x \in H$

$$(V_\lambda x, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda t} |x(t)|^2 dt,$$

$$\operatorname{Re}(V_\lambda x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \lambda t \cdot |x(t)|^2 dt.$$

It follows easily that, if λ is real and not an integer and if the numbers x_n are real and vary under the condition $\sum_1^\infty x_n^2 = 1$, then the upper and lower bounds of the quadratic form

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m x_n}{m-n+\lambda}$$

are $\pi \operatorname{cosec} \pi\lambda$ and $\pi \operatorname{cotg} \pi\lambda$ when $|\lambda| < 1$, and are

$$\pm \pi |\operatorname{cosec} \pi\lambda|$$

when $|\lambda| > 1$; these bounds are not attained.

B. Sz.-Nagy (Szeged).

Bari, N. K. Biorthogonal systems and bases in Hilbert space. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 69–107 (1951). (Russian)

Let $S = \{f_n; g_n\}$ be a biorthonormal (BN) system in L_2 ($(f_n, g_m) = \delta_{nm}$) for which both the sets $\{f_n\}$ and $\{g_n\}$ are assumed complete. $\{f_n\}$ is called a Bessel system in case $f \in L_2$ implies $\sum_{n=1}^{\infty} |(f, g_n)|^2 < \infty$; $\{f_n\}$ is called a Hilbert system in case $\sum_{n=1}^{\infty} |C_n|^2 < \infty$ implies there is an f in L_2 for which $(f, g_n) = C_n$; $\{f_n\}$ is called a Fischer-Riesz system in case it is both a Bessel and a Hilbert system. Typical theorems relating these concepts are: 1) $\{f_n\}$ is a Bessel system if and only if there is a complete orthonormal (CON) system $\{h_n\}$ in L_2 and a bounded linear operator A in L_2 for which $Af_n = h_n$. 2) $\{f_n\}$ is a Hilbert system if and only if there is a CON system $\{h_n\}$ in L_2 and a bounded linear operator C in L_2 for which $Ch_n = g_n$. 3) $\{f_n\}$ is a Bessel system if and only if $\{g_n\}$ is a Hilbert system. 4) $\{f_n\}$ is a Bessel system if and only if there is a bounded, positive Hermitian operator B in L_2 for which $g_n = Bf_n$. 5) If $\{f_n\}$ is a Fischer-Riesz system, then there are two positive constants, m, M such that for any f in L_2,

$$m^2 \|f\|^2 \leqq \sum_{n=1}^{\infty} |(f, g_n)|^2 \leqq M^2 \|f\|^2.$$

6) $\{g_n\}$ is a Fischer-Riesz system if and only if there is an invertible bounded linear operator A in L_2 and a CON system $\{h_n\}$ for which $Af_n = h_n$. 7) $\{f_n\}$ is a Fischer-Riesz system if and only if there is an invertible, positive, bounded Hermitian operator B for which $f_n = Bg_n$.

Examples are given of bases in Hilbert space which are not Fischer-Riesz bases. In accordance with the result of Gelfand [see the following review] these must be non-absolute bases in L_2, i.e., there exist non-absolute bases in L_2. It is to be observed that this last result was obtained independently by the reviewer [Proc. Amer. Math. Soc. **2**, 720–721 (1951); these Rev. **13**, 253]. *B. R. Gelbaum.*

Gel'fand, I. M. Remark on the work of N. K. Bari, "Biorthogonal systems and bases in Hilbert space." Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 224–225 (1951). (Russian)

In the paper of Bari reviewed above, the notion of equivalence of bases is defined thus: Two bases (or systems) $\{f_n\}$ and $\{\tilde{f}_n\}$ are equivalent if there is a bounded invertible operator A in L_2 for which $f_n = A\tilde{f}_n$. Recall that a basis $\{f_n\}$ is called absolute if $\{f_{n_k}\}$ is also a basis for each $1-1$ map, n_k, of the integers onto themselves. In this note the author shows that a basis is absolute if and only if it is equivalent to an orthogonal (hence ON) basis.

B. R. Gelbaum (Minneapolis, Minn.).

Heinz, Erhard. Ein v. Neumannscher Satz über beschränkte Operatoren im Hilbertschen Raum. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. Math.-Phys.-Chem. Abt. **1952**, 5–6 (1952).

Using the algorithm of I. Schur, von Neumann has shown that if A is a bounded linear operator in Hilbert space with norm $|A| < 1$ and if $f(z)$ is an analytic function for $|z| < 1$ and for such z, $|f(z)| \leqq 1$, then the norm $|f(A)| \leqq 1$. This theorem is proved in the present note by means of the Poisson integral which is used to prove the following interesting result. If for $|z| < 1$, $f(z)$ is analytic and $\operatorname{Re} f(z) \geqq 0$ and if A is a linear bounded operator in Hilbert space H with norm $|A| < 1$, then for every φ in H, $\operatorname{Re}(f(A)\varphi, \varphi) \geqq 0$. It is also shown that if B is a bounded linear operator with $\operatorname{Re}(B\varphi, \varphi) \geqq 0$, $\varphi \in H$, then $(B+I)^{-1}$ exists and $|(B-I)(B+I)^{-1}| \leqq 1$. *N. Dunford.*

Rabinovič, Yu. L. On the continuous dependence upon a parameter of the spectrum of a symmetric linear integral operator. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 181–191 (1951). (Russian)

The author considers a symmetric operator K_ϵ of Hilbert-Schmidt type in Hilbert space, depending on a parameter ϵ in an interval (ϵ_1, ϵ_2). If $\|K\|$ denotes the Hilbert-Schmidt norm of K, it is assumed that $\|K_\epsilon\|$ is uniformly bounded, and that K_ϵ is continuous in norm as a function of ϵ. Under these assumptions, the author proves (among others) the following results. (a) If $\epsilon_1 < \epsilon_0 < \epsilon_2$, and $\mu(\epsilon_0)$ is a non-zero

eigen-value of K_{ϵ_0}, then there is a continuous function $\mu(\epsilon)$, defined in some neighbourhood of ϵ_0, such that $\mu(\epsilon)$ is an eigen-value of K_ϵ. (b) If $\{\mu_n^+(\epsilon)\}$ is the sequence of positive eigen-values of K_ϵ, repeated according to multiplicity, and written in decreasing order of magnitude, and $\{\mu_n^-(\epsilon)\}$ is the similarly arranged sequence of negative eigen-values, then $\mu_n^+(\epsilon)$ and $\mu_n^-(\epsilon)$ are continuous functions of ϵ (an infinite sequence of 0's being adjoined to either sequence if necessary). (c) Write $E(\epsilon)$ for the projector on the eigen-subspace corresponding to the eigen-value $\mu(\epsilon)$. Suppose that $\mu(\epsilon_0)$ has multiplicity ν with respect to ϵ, in the sense that when $\epsilon \to \epsilon_0$, there are ν eigen-values of K_ϵ (denoted by $\mu_1(\epsilon), \cdots, \mu_\nu(\epsilon)$) which converge to $\mu(\epsilon_0)$. Then

$$E_1(\epsilon)+\cdots+E_\nu(\epsilon)\to E(\epsilon_0) \quad (\epsilon\to\epsilon_0),$$

convergence being in the norm sense.

The main tool used in the proofs is the consideration of the properties of the expression

$$q_0(\mu, \epsilon)=\inf\{\|\mu f-K_\epsilon f\|^2: \|f\|=1\}.$$

For instance, if $\mu\neq 0$, and there is no eigen-value of K_ϵ between 0 and 2μ, then $q_0(\mu, \epsilon)=\mu^2$; on the other hand, if such an eigen-value exists, and the nearest one to μ is denoted by $\mu^0(\epsilon)$, then (1) $\mu^0(\epsilon)=\mu\pm(q_0(\mu, \epsilon))^{1/2}$. Since $q_0(\mu, \epsilon)$ is a continuous function of the pair (μ, ϵ), the relation (1) proves extremely useful. *F. Smithies* (Cambridge, England).

Vaĭnberg, M. M. On invariant directions of the product of certain operators. Doklady Akad. Nauk SSSR (N.S.) **85**, 261–263 (1952). (Russian)

Let B be a self-adjoint, completely continuous operator on a Hilbert space H to itself and let F be a potential operator from H into itself; i.e., F is a gradient of some functional. Let W be the set of non-zero $x \in H$ with direction fixed under BF, that is, such that there exists a $\mu\neq 0$ with $BF(x)=\mu x$. The author states two theorems about the nature of W. For example, if F is continuous on H, if $F(-x)=-F(x)$, and if $(F(x), x)>0$ when $x\neq 0$, then, on each sphere $\|x\|=c>0$, W contains a countable set of linearly independent elements which can be represented by suitable series in terms of the characteristic numbers and vectors of B. The theorems apply to certain non-linear integral equations. [See Vaĭnberg, Mat. Sbornik N.S. **30(72)**, 3–10 (1952); these Rev. **13**, 658.] *M. M. Day.*

Weinberger, H. F. An optimum problem in the Weinstein method for eigenvalues. Pacific J. Math. **2**, 413–418 (1952).

Let L be a completely continuous positive symmetric operator in the Hilbert space H, and $\lambda_1\geqq\lambda_2\geqq\cdots$ its eigenvalues. Let G be a subspace of H, and $\lambda'_1\geqq\lambda'_2\geqq\cdots$ be the eigenvalues of the projection L' of L into G. Finally, let $p_1, \cdots, p_m$ be m arbitrary elements of the orthogonal complement $P=H\ominus G$ of G, and $\lambda_n^{(m)}=\lambda_n^{(m)}(p_1, \cdots, p_m)$ the eigenvalues of the projection of L into $H\ominus\{p_1, \cdots, p_m\}$. Then the inequalities

$$\lambda_n^{(m)}\geqq\lambda'_n, \quad \lambda_n^{(m)}\geqq\lambda_{n+m} \tag{1}$$

are known. The object of the paper is the proof of the following theorem: For fixed n, m the $p_1, \cdots, p_m$ can be chosen in such a way that the weaker of the inequalities(1) becomes an equality. *E. H. Rothe* (Ann Arbor, Mich.).

Weinberger, H. F. Error estimation in the Weinstein method for eigenvalues. Proc. Amer. Math. Soc. **3**, 643–646 (1952).

Let L be a positive completely continuous operator in the Hilbert space H, and let u_n and λ_n be the eigenvectors and eigenvalues of L. Let G be a subspace of H, L' the projection of L into G, and λ'_n the eigenvalues of L'. Let $(p_1, p_2, \cdots)$ be a complete set of the orthogonal complement $P=H\ominus G$ of G, and $\lambda_n^{(m)}$ be the eigenvalues of the projection of L into $H\ominus\{p_1, \cdots, p_m\}$. Then it is known that $\lambda_n^{(m)}\geqq\lambda_n'$ and that $\lim_{m\to\infty}\lambda_n^{(m)}=\lambda'_n$. In the present paper, the estimate

$$\lambda_n^{(m)}-\lambda'_n\leqq\lambda_{m+1} \tag{1}$$

is proved if for p_n the projection of u_n into P is chosen. This estimate is uniform in the eigenvalues (i.e., it does not depend on n), and depends only on the eigenvalues of L. It is shown that the estimate (1) is the best one with these two properties. Let, finally, S be some subspace of H containing P, and λ_n'' be the eigenvalues of the projection L'' of L into S. Then the estimate $\lambda_n^{(m)}\leqq\lambda_n\leqq\lambda_n^{(m)}+\lambda''_{m+1}$ for the λ_n is given. *E. H. Rothe* (Ann Arbor, Mich.).

Takenouchi, Osamu. On the maximal Hilbert algebras. Tôhoku Math. J. (2) **3**, 123–131 (1951).

Takenouchi, Osamu. On the structure of maximal Hilbert algebras. Math. J. Okayama Univ. **1**, 1–32 (1952).

A Hilbert algebra is an algebra A over the complex field which is a dense linear subspace of a Hilbert space H such that: 1) the linear operations in A and H coincide; 2) to each $x \in A$ there exists an $x^* \in A$, characterized by the fact that in A left and right multiplications by x are adjoint to left and right multiplications by x^*; 3) the operation of left multiplication by any $x \in A$ is bounded on A and so can be extended to a bounded operator, denoted by L_x, on H; 4) if $L_x y=0$ for all $x \in A$, then $y=0$. If $y \in H$, then operators L_y^0 and R_y^0, with domain A, are defined by: $L_y^0 x=R_x y$, $R_y^0 x=L_x y$. Hilbert algebras in H which are extensions of A are considered and it is shown there exists a unique maximal extension consisting of those $y \in H$ such that L_y^0 or R_y^0 is bounded. It is shown that if A is maximal, then H can be decomposed into closed linear subspaces H_λ such that, letting $A_\lambda=H_\lambda\cap A$, A_λ is a maximal algebra in H_λ, A_λ is an ideal in A, and each A_λ is either simple or contains no simple ideals. If A is simple, then the rings of left and right multiplications are coupled factors in the sense of Murray and von Neumann. A theorem is proved reducing certain considerations to the case when A has an identity by representing A as a matrix whose elements are from Hilbert algebras containing an identity. In the separable case it is proved that if A is maximal and contains no simple ideals, then it can be decomposed into a direct integral of maximal Hilbert algebras. *W. Ambrose* (Cambridge, Mass.).

Buck, R. Creighton. Operator algebras and dual spaces. Proc. Amer. Math. Soc. **3**, 681–687 (1952).

G denoting a topological group, let $C^*(G)$ be the linear space of all complex continuous functions on G, U_x ($x \in G$) the left translation operators on $C^*(G)$, and X a subspace of $C^*(G)$ provided with a locally convex topology τ. Let $L(X)$ and $B(X)$ be, respectively, the spaces of linear functionals and linear operators on X whose restrictions to the τ-bounded sets are τ-continuous. If X is invariant under translations, and if τ is suitable, the U_x belong to $B(X)$; in this case $B^0(X)$ will denote the subalgebra of $B(X)$ formed of those T in $B(X)$ which commute with translations. For suitable choices of X and τ, the mapping which assigns to $T \in B^0(X)$ the functional F defined by $F(\phi)=T(\phi)(0)$ defines a linear space isomorphism between $B^0(X)$ and $L(X)$, the inverse of which assigns to $F \in L(X)$ the operator T defined by $T(\phi)=\psi$ with $\psi(x)=F(U_x\phi)$. Ordinary operator

multiplication in $B^0(X)$ then induces a convolution process in $L(X)$ under which the latter becomes an algebra isomorphic to $B^0(X)$. Further, defining the norms $\|F\|=\sup|F(\phi)|$ and $\|T\|=\sup\|T(\phi)\|_\sigma$ for ϕ in X and $\|\phi\|_\sigma=\sup_{x\in\sigma}|\phi(x)|\leqq 1$, it is shown (Theorem 2) that $L(X)$ is a topological algebra under the convolution and the norm $\|\cdot\|$ for three specified choices of X and τ.

On specialising G to the additive group of reals, contact is made with the theory of distributions. Thus, for example, it is shown that any endomorphism of the space $\mathcal{S}$ of all indefinitely differentiable functions provided with the topology defined by the semi-norms $\|\phi\|_{K,p}=\sup_{x\in K}|(d/dx)^p\phi(x)|$ ($p=0, 1, 2, \cdots$; K a compact subset of G) takes the form of convolution with a distribution having a compact support [cf. L. Schwartz, Théorie des distributions, t. II, Hermann, Paris, 1951, Théorème X, p. 18; these Rev. 12, 833].

The basic idea behind the article appears to be similar to that proposed by Hewitt and Zuckerman [Proc. Internat. Congress Math., v. I, Cambridge, Mass., 1950, Providence, R. I., 1952, p. 455].

R. E. Edwards (Nancy).

Kaplansky, Irving. A theorem on rings of operators. Pacific J. Math. 1, 227–232 (1951).

The main result of the paper is the following. Let M and N be self-adjoint algebras of bounded operators on Hilbert space. Suppose that $M\subset N$ and that M is strongly dense in N. Then the unit sphere of M is strongly dense in the unit sphere of N. The proof begins with the study of continuity properties of functions $f(A)$ on certain subsets of the set of normal operators (lemmas 1–5). Then the unit sphere Z_1 in the set of self-adjoint elements of M is considered and it is shown that any self-adjoint element B of N with bound $\leqq 1$ is in the closure of Z_1. The final transition from self-adjoint to arbitrary elements of N is made by considering the algebra of all 2×2 matrices over N. Another result (Theorem 2) of the paper is concerned with a continuity property of the mapping $A\to h(A)$ on self-adjoint elements A, defined by a bounded real-valued Baire function $h(t)$ of a real variable t.

F. I. Mautner (Baltimore, Md.).

Kaplansky, Irving. Algebras of type I. Ann. of Math. (2) 56, 460–472 (1952).

This is a continuation of an earlier paper [Ann. of Math. (2) 53, 235–249 (1951); these Rev. 13, 48], and the same terminology is used. The main object of study of the present paper is the structure of AW^*-algebras of type I. And results which are known for weakly closed self-adjoint algebras of operators on Hilbert space are proved under the weaker assumptions embodied in the definition of an AW^*-algebra. For example, one of the main results is that an AW^*-algebra is uniquely determined by its center, provided that the given algebra is homogeneous in a certain sense. The following results should also be noted. An AW^*-algebra of type I is weakly closed if and only if its center is. Any *-automorphism of an AW^*-algebra which leaves the center elementwise fixed is inner by a unitary element.

F. I. Mautner (Baltimore, Md.).

Calculus of Variations

De Giorgi, Ennio. Compiuta ricerca dell'estremo inferiore di un particolare funzionale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 [1951]. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 334, 13 pp. (1952).

The author studies in detail the variable end point problem for functionals of the form

$$I[x]=\int_{t_1}^{t}(px^2+qx^2+2rx)dt,$$

stating that it leads to "interesting and a priori unexpected circumstances." He gives examples. *J. M. Danskin.*

Kerimov, M. K. On sufficient conditions for an extremum in discontinuous variational problems with variable endpoints. Doklady Akad. Nauk SSSR (N.S.) 84, 213–216 (1952). (Russian)

For the problem of calculus of variations, for necessary conditions for an extremum and for the notations we refer to two previous papers [Kerimov, same Doklady 79, 565–568, 719–722 (1951); these Rev. 13, 474]. The methods of Bliss are used in the present paper. The Weierstrass, Legendre, and Jacobi conditions in their weak or strong forms are proved to be sufficient in order that a curve E_{123} gives a weak or strong relative minimum for the functional J in the class of all curves C_{123}.

L. Cesari (Lafayette, Ind.).

Sigalov, A. G. On conditions of differentiability and analyticity of solutions of two-dimensional problems of the calculus of variations. Doklady Akad. Nauk SSSR (N.S.) 85, 273–275 (1952). (Russian)

Let $F(x, y, z, p, q)$ be a continuously twice differentiable real function of its real variables defined for all z, p, q and for (x, y) in some Jordan domain D. It is convenient to group F and its derivatives of orders $\leqq 2$ into five sets (a), (b), (c), (d), (e), where (a) consists of F, F_z, (b) of F_p, F_q, F_{zz}, F_{zq}, (c) of F_{pz}, F_{qz}, F_{pp}, F_{qq}, (d) of the partial derivatives of order 2 in (z, p, q), and (e) of the remainder, the derivatives in (x, y) which we disregard. Further, $F_1(x, y, z, p, q)$ shall denote the minimum for $u^2+v^2=1$ of the quadratic form $F_{pp}u^2+2F_{pq}uv+F_{qq}v^2$. The author now makes the following assumptions concerning F. For each $R>0$ there exist positive M, L, m depending on R, such that, for (x, y) in D and $|z|\geqq R$: 1) the terms of (d) are absolutely $\leqq M$; 2) those of (a), the squares of those of (b), and for some $k>2$ the kth powers of those of (c) are absolutely $\leqq M(p^2+q^2)$; 3) for $p^2+q^2\geqq L^2$ we have $F_1\geqq m$. This being so, let $z(x, y)$ provide the absolute minimum of $\iint F dxdy$ over D in the class of functions which agree with it on the rim of D and are ACT and of finite Dirichlet integral. The author's main assertion, in whose proof he extends a procedure of Shiffman [Ann. of Math. (2) 48, 274–284 (1947); these Rev. 9, 45] is that $z(x, y)$ is Lipschitzian in every closed subdomain of D. If in addition the second derivatives of F satisfy a Hölder condition, or if F is analytic, then $z(x, y)$ is twice differentiable, or else analytic; this now follows from known results of C. B. Morrey [Trans. Amer. Math. Soc. 43, 126–166 (1938)] and S. Bernstein [Math. Ann. 59, 20–76 (1904)].

L. C. Young (Madison, Wis.).

Danskin, John M., Jr. On the existence of minimizing surfaces in parametric double integral problems of the calculus of variations. Rivista Mat. Univ. Parma **3**, 43–63 (1952).

Cesari, Lamberto. An existence theorem of calculus of variations for integrals on parametric surfaces. Amer. J. Math. **74**, 265–295 (1952).

The theorem proved by each of the two authors was announced by Danskin at the end of 1949 and by Cesari in 1950; a preliminary version is given in Danskin's Ph.D. thesis of July, 1949. However the first complete published proof is due to Sigalov [Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 16–101 (1951); these Rev. **13**, 257] who also published a previous abbreviated version, containing a minor gap, in two notes [Doklady Akad. Nauk SSSR (N.S.) **70**, 769–772; **71**, 617–620 (1950); these Rev. **11**, 603, 604]. It is now possible for the first time to compare the three proofs, all of which use the well-known semi-continuity theory [cf. McShane, Ann. of Math. (2) **33**, 460–484 (1932)], so that their main task is to establish the existence of a suitably compact minimizing sequence of parametric surfaces. In the case of curves, the corresponding compactness problem had been solved long before semi-continuity was studied; however, for surfaces it is a very much deeper problem and the available results dealt with one or other of two modified forms of compactness obtained, on the one hand, by J. W. Calkin [Duke Math. J. **6**, 170–186 (1940); these Rev. **1**, 208] and C. B. Morrey [ibid. **6**, 187–215 (1940); also Univ. California Publ. Math. (N.S.) **1**, 1–130 (1943); these Rev. **1**, 209; **6**, 180], and, on the other hand, by the reviewer [Trans. Amer. Math. Soc. **64**, 317–335 (1948); these Rev. **10**, 109]. Both these forms of compactness were, however, inadequate in one respect—and unavoidably so because they made no use of any minimizing assumption: they failed to ensure continuity of a limit surface. The crucial problem which faced Danskin, Sigalov, and Cesari was thus precisely to use the minimizing assumption for the given sequence of surfaces to transform the latter suitably into a sequence with a continuous limit. Their solutions of this crucial problem are quite different from one another.

Danskin simply applies a continuity criterion due to C. B. Morrey. However, since he uses the form of compactness studied by C. B. Morrey and J. W. Calkin, his minimizing sequence converges to this continuous limit only in an appropriate Hilbert space, and his main effort has to be directed to modifying this sequence so as to make it converge in the ordinary sense to the same limit, without losing its minimizing property. This is, in effect, equivalent to restating the semi-continuity theory for a new type of convergence. It is clear from the reviews cited at the beginning, that this procedure is quite different from that of Sigalov.

Cesari's solution, which follows the same general lines, is again very different in detail from the other two. It is longer, and perhaps slightly later, than they; however, it is most remarkable for the finished character of two of its tools (which have an interest quite apart from the result established by their means) and for its elementary character. One tool, a retraction theorem established in a second paper [see the following review] which is really part of a complete presentation, enables him to reduce everything to polyhedra. The other tool, which concerns polyhedra, constitutes a beautiful refinement of a lemma of the reviewer [J. London Math. Soc. **19**, 209–212 (1944); these Rev. **7**, 197] and enables him to overcome the crucial problem of continuity of the limit surface.

The work of Danskin is limited to a symmetric integrand. This temporary restriction is not made by the other two authors. *L. C. Young.*

Cesari, Lamberto. Su un particolare processo di retrazione per superficie. Rivista Mat. Univ. Parma **3**, 25–42 (1952).

In the sequel E is Euclidean 3-space and R is the closure of a fixed plane Jordan domain. A continuous vector-function $x(u, v)$, with values in E and which is defined and not everywhere constant for (u, v) in R, is termed of type A if no maximal subcontinuum W of R of constancy of $x(u, v)$ disconnects the (u, v)-plane, and is termed non-degenerate if further no such W disconnects R. A continuous $y(u, v)$ defined in R is termed retraction of $x(u, v)$ if there is a continuum $K \subset R$ such that $y(u, v)$ coincides with $x(u, v)$ on K and is constant in each component of $R-K$. Further, the closed Fréchet curve described by a continuous $x(u, v)$ on the rim of R will be termed boundary curve of $x(u, v)$. This being so, the author's main result is as follows: Given a simple closed curve C in E and given $a>0$, there exists a number $b=b(C, a)>0$ such that every $x(u, v)$ of type A, whose boundary curve has Fréchet distance $<b$ from C, has a non-degenerate retraction $y(u, v)$ whose boundary curve has Fréchet distance $<a$ from C. *L. C. Young.*

Magenes, Enrico. Sul minimo relativo nei problemi di calcolo delle variazioni d'ordine n. Rend. Sem. Mat. Univ. Padova **21**, 1–24 (1952).

The problem considered is that of minimizing an integral of the form

$$I(C)=\int_{x_1}^{x_2} f(x, y, y', y'', \cdots, y^{(n)})dx$$

in a class of "ordinary curves" in the plane satisfying suitable end conditions. An ordinary curve, C: $y=y(x)$, is one for which the defining function $y(x)$ is of class $C^{(n-1)}$ and has an absolutely continuous $(n-1)$th derivative, and which gives $I(C)$ a finite value. A lagrangian neighborhood of order q $(0 \leqq q \leqq n)$ of a curve C_0: $y=y_0(x)$ of class $C^{(n)}$ consists of all ordinary curves C satisfying $|y^{(i)}(x)-y_0^{(i)}(x)| \leqq \rho$ for $i=0, \cdots, q$, $x_1 \leqq x \leqq x_2$. The neighborhoods of order q determine the topology of order q and the meaning of a relative minimum of order q for $I(C)$ in a natural way. Known theorems give sufficient conditions that a curve C_0 (of class $C^{(n)}$) shall furnish a relative minimum of order n (weak relative minimum) or a relative minimum of order $n-1$ (strong relative minimum). Magenes obtains sufficient conditions for a relative minimum of order zero by adding to the conditions for a strong relative minimum the following: There exists a neighborhood N_0 of order zero of C_0 and a function $\Phi(r)$ such that $f_{y^{(n)}y^{(n)}} \geqq 0$ on N_0, $f \geqq \Phi(|y^{(n)}|)$ on N_0, where $\lim_{r \to +\infty} \Phi(r)/r = +\infty$, and $\Phi(r)$ is bounded below for $r \geqq 0$. The proof depends on a theorem of Cinquini on the existence of an absolute minimum for $I(C)$. A modified set of sufficient conditions appears in Theorem 2, the proof for which is not given. The author seems to be in error in his statement on p. 23 to the effect that a class of ordinary curves which is closed in the topology of order $q<n-1$ need not be closed in the topology of order $n-1$. *L. M. Graves* (Chicago, Ill.).

***Bers, Lipman. Singularities of minimal surfaces.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 157–164. Amer. Math. Soc., Providence, R. I., 1952.

A report on the author's work on solutions of the equation of minimal surfaces with isolated singularities [cf. L. Bers,

Ann. of Math. 53, 364–386 (1951); Trans. Amer. Math. Soc. 70, 465–491 (1951); J. Analyse Math. 1, 43–58 (1951); these Rev. 13, 244, 245, 244]. In the last section the results obtained for the minimal surface equation are compared with results for linear equations and some generalizations are conjectured, one of which has been since verified by R. S. Finn. *R. Courant* (New York, N. Y.).

Theory of Probability

*Gnedenko, B. V., i Hinčin, A. Ya. **Èlementarnoe vvedenie v teoriyu veroyatnostei.** [**Elementary introduction to the theory of probability.**] 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, Leningrad, 1952. 144 pp. 2.20 rubles.

Kawada, Yukiyosi. On the fundamental notions of the probability theory. Math. Japonicae 2, 103–116 (1952).

The author makes the mathematical basis of probability a Boolean algebra together with a non-negative additive function on the algebra, with maximum value 1. A random variable is a set of elements $\{E_a, -\infty<a<\infty\}$ in the algebra, with E_a monotone, and such that $p(E_a)$ defines a probability distribution function. Families of random variables are defined analogously. These can be mapped on families of random variables of the usual type (measurable functions defined on a measure space), and the standard concepts like summation, convergence, and so on can then be reduced to these concepts as applied to the image families, where they are classical. *J. L. Doob* (Urbana, Ill.).

Hinčin, A. Ya. On classes of equivalent events. Doklady Akad. Nauk SSSR (N.S.) 85, 713–714 (1952). (Russian)

A sequence of equivalent events was defined by Finetti as a sequence of events with the property that the probability of occurrence of any number r of the events is a number ω_r depending only on r, not on the chosen events. A sequence of independent events with a common probability p (Bernoulli trials) is a class of equivalent events, as is the more general sequence obtained from this by allowing p to be a random variable. In the latter case, $\omega_r=\int_0^1 x^r dP(x)$, where P is the distribution function of p, and on the other hand the author has shown [Mat. Sbornik 39, no. 3, 40–43 (1932)] that this formula with an arbitrary distribution function P gives the most general ω_r sequence. Thus the two situations can be identified with each other, and the theory of Bernoulli trials can be used to prove the theorems on sequences of equivalent events. *J. L. Doob.*

Consael, R. Sur les processus de Poisson du type composé. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 442–461 (1952).

In the first part of the paper the author systematically discusses compound Poisson processes

$$P_n(t)=\int e^{-tx}\frac{(tx)^n}{n!}dU_x,$$

where $P_n(t)$ is the probability of n successes in the time interval $(0, t)$ and $U(x)$ is a cumulative distribution. Explicit formulas are derived for $P_n(t)$ and the moments of n when dU_x is the Poisson, Pascal, the Pólya-Eggenberger, and the Pearson Type III probability functions. The author next takes up the problem of the construction of processes which are compound Poisson processes and generalized Poisson processes at the same time. Let two independent variables X and Y be distributed respectively according to $G_i(x)=\int_{-\infty}^{\infty}F(x, a)dU_i(a)$, $i=1, 2$, where $U_i(a)$ is a cumulative distribution function, and let $F(x, a)$ denote another cumulative distribution function such that $F(x, u+v)$ is the convolution of $F(x, u)$ and $F(x, v)$; then $X+Y$ is distributed according to the same law $G(x)$ associated with a cumulative distribution function $U(a)$, the convolution of $U_1(a)$ and $U_2(a)$. In case $F(x, a)$ is the Poisson distribution function this theorem reduces to that of Feller [Ann. Math. Statistics 14, 389–400 (1943); these Rev. 5, 209]. The theorem is applied to the formation of probability functions exhibiting the property of being a compound and generalized Poisson process at the same time. *L. A. Aroian.*

Nisida, Tosio. On the inverse function of Poisson process. Math. Japonicae 2, 135–142 (1952).

Let $T(n)$ be the time it takes for n events to occur in a Poisson process. Then $T(n)$ is the sum of n mutually independent random variables with a common exponential density. This familiar fact is carefully proved, along with some results that follow from it. *J. L. Doob.*

Varma, R. S. On the probability function in a normal multivariate distribution. Quart. J. Mech. Appl. Math. 5, 361–362 (1952).

The author derives three formulas suitable for the numerical calculation of

$$Z_n(z, a)=\frac{1}{\Gamma(\frac{1}{2}n-\frac{1}{2})2^{n/2-1}\pi^{1/2}}\int_0^z\int_0^\pi t(t\sin\theta)^{n-2}\times\exp\{-\tfrac{1}{2}(t^2+a^2-2at\cos\theta)\}d\theta dt,$$

which was evaluated otherwise by Quenouille [Proc. Edinburgh Math. Soc. (2) 8, 95–100 (1949); these Rev. 11, 673]. *L. A. Aroian* (Culver City, Calif.).

Banerjee, D. P. On some inequalities in the theory of probability. Ganita 2, 53 (1951).

As is well known, Liapounoff's inequality in probability theory is a simple instance of Hölder's inequality. [The reviewer could never understand why so many "elementary" proofs were published.] The present note takes cognizance of Hölder's and Minkowski's inequalities. *K. L. Chung.*

Ivanović, B. Précision de la déviation standarde pour une répartition quelconque. Srpska Akad. Nauka. Zbornik Radova 18, Matematički Inst. 2, 173–180 (1952). (Serbo-Croatian. French summary)

It is shown that Chebyshev's inequality may lead to a more precise estimate than the corresponding inequality which uses a higher moment. *W. Feller* (Stockholm).

Pompilj, G. Sulla media e la varianza di un campione. Gaz. Mat., Lisboa 12, no. 50, 69–75 (1951).

Pour les H individus $e_1, e_2, \cdots, e_H$ d'une population P, un caractère X prend respectivement les valeurs $x_1, x_2, \cdots, x_H$ distinctes ou non; soient a) $X_1, \cdots, X_N$ les valeurs de X pour N individus extraits au hasard et en grappe de P; b) $x_1', x_2', \cdots, x_K'$ les valeurs distinctes des x_j $(K\leqq H)$; c) Z la variable aléatoire multi-dimensionnelle $(Z_1, Z_2, \cdots, Z_K)$ où Z_k est le nombre des X_j qui valent x_k' $(\sum Z_k=N)$; en s'appuyant sur en étude de Z, l'auteur étudie les variables aléatoires (moments empiriques) $\sum_{j=1}^{N}X_j^r/N$, et en particulier calcule explicitement leurs espérances mathématiques et leurs variances pour $r=1$ et 2. *R. Fortet.*

Morimura, Hidenori. A remark on the dispersion of sums of independent random variables. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 3, 1–5 (1951).

Let $F(x)$ be a distribution function with maximal concentration function $Q(F;l)=\sup(F(x+l+0)-F(x))$. Denote by $D_F(\alpha)$ the inverse of $Q(F;l)$. Let x_k be a sequence of mutually independent random variables with distribution F_k and variance σ_k^2 and such that $\lim P\{|x_k|\geqq nD_n\}=0$, $1\leqq k\leqq n$, uniformly in k. Let $V_n^2=\sum_{j=1}^n\sigma_j^2$. Then the author proves: A necessary and sufficient condition for the existence of constants K, K' such that $K\leqq V_n/D_n(\alpha)\leqq K$ for all n, is that

$$\sum_{k=1}^{n}\left\{\int_n^\infty x(1-F_k(D_n(\alpha)x)dx-\int_{-\infty}^{-n}xF_k(D_n(\alpha)x)dx\right\}$$

are uniformly bounded for $n=1, 2, \cdots$, where $0<\alpha<1$ and K and K' are dependent only on α. A similar relation between the variance and inverse of the mean concentration function for sums of independent random variables was proved by Kunisawa [Kōdai Math. Sem. Rep. 1951, 71–72; these Rev. 13, 566]. *J. L. Snell* (Princeton, N. J.).

García Alvarez, M. Continuous probabilities. Revista Mat. Hisp.-Amer. (4) 11, 243–256 (1951). (Spanish)

The main purpose of this expository article is to describe certain known results on so-called geometrical probabilities. *P. R. Halmos* (Chicago, Ill.).

Tortrat, A. Sur la divisibilité des lois convexes de probabilité. Bull. Soc. Math. France 80, 47–60 (1952).

If $y(t)$ is a real, continuous, even function which is convex for $t>0$ and which equals one at $t=0$ and vanishes at infinity, then it is well known that $y(t)$ is a characteristic function (c. f.). The author calls such a function a convex c. f. and studies the decomposability of a convex c. f. into the product of two convex c. f.'s. Elementary geometry and calculus are used. The main result, a necessary and sufficient condition (whether or not under some additional hypothesis is not quite clear to the reviewer), is geometrical in nature and rather complicated. *K. L. Chung* (Ithaca, N. Y.).

Kunisawa, K., and Maruyama, G. Some properties of infinitely divisible laws. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 3, 22–27 (1951).

Let X be distributed according to an infinitely divisible law and let $X=X_{n1}+\cdots+X_{nn}$ where the X_{nk} are independent and identically distributed. Then $Y_n=X_{n1}^2+\cdots+X_{nn}^2$ converges in law to an infinitely divisible law whose canonical representation is simply related to that of X. In particular, Y_n converges in probability to a constant if and only if X is normally distributed. This is closely related to a result of Raikov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 1938, 323–338]. Two other corollaries are deduced, dealing respectively with the Poisson law and the limiting distribution of $\max_k X_{nk}$. In Theorem 4 read $\exp(-cx^{-\alpha})$ for $\exp(-x^{-\alpha})$ where c is a positive constant. *K. L. Chung.*

Udagawa, Masatomo. On numbers of positive sums of independent random variables. Kōdai Math. Sem. Rep. 1952, 45–50 (1952).

Let $\{x_n\}$ be a sequence of independent random variables. Let S_n be the sum of the first n x_j's and N_n the number of S_j's, $1\leqq j\leqq n$, which are positive. The arc sine law is said to hold if $\lim \Pr\{N_n/n<\alpha\}=2\pi^{-1}\arcsin\alpha^{\frac{1}{2}}$, $0\leqq\alpha\leqq 1$. Andersen [Skand. Actuarietidskr. 32, 27–36 (1949); these Rev. 11, 256] proved that the arc sine law holds for identically symmetrically distributed independent random variables. Erdös and Kac proved [Bull. Amer. Math. Soc. 53, 1011–1020 (1947); these Rev. 9, 292] that the arc sine law holds for independent random variables having mean 0 and variance 1 and satisfying the central limit theorem. In this paper the arc sine law is proved for identically distributed random variables assuming only that S_n/A_n approaches the normal distribution for some positive constants A_n approaching infinity. The method of proof follows that of Erdös and Kac, replacing variance arguments using Kunisawa's mean concentration function. A second proof is given assuming only that S_n/A_n approaches a symmetric stable distribution.

J. L. Snell (Princeton, N. J.).

Maruyama, Gisirō. Note on the arc sine law in the theory of probability. Nat. Sci. Rep. Ochanomizu Univ. 2, 25–27 (1951).

Let $\{x_k\}$ be a sequence of independent random variables having symmetric distribution laws and let S_n be the sum of the first n of the x_k's. Assume that $S_n/\log n$ approaches a normal distribution. The author shows that with these norming constants the arc sine law [see above review] does not hold and proves that:

$$P\left\{\frac{1}{\log n}\sum_{\nu=1}^{n}\left(\frac{N_\nu}{\nu}\right)\frac{1}{\nu}\leqq x\right\}\to\frac{2}{\pi}\arcsin\sqrt{x},\quad 0\leqq x\leqq 1,$$

where N_ν is the number of positive S_k, $1\leqq k\leqq\nu$.

J. L. Snell (Princeton, N. J.).

Hammersley, J. M. An extension of the Slutzky-Fréchet theorem. Acta Math. 87, 243–257 (1952).

If $\{x_n\}$ is a sequence of (vector) random variables converging in distribution to a random variable x, the author writes $\operatorname{dlim} x_n=x$. In what sense is it true that if $\operatorname{dlim} x_n=x$, then (1) $\operatorname{dlim}(x_n-x)=0$ and (2) $\operatorname{dlim} y(x_n)=y(x)$, where y belongs to a suitably restricted class of functions? One answer to the first question is that there exists, for each n, a joint distribution of the pair (x_n, x) with the property that its marginal distributions are the given distributions of x_n and of x, and with respect to which $\operatorname{dlim}(x_n-x)=0$. The author offers also an answer to the second question. That answer is formulated in terms of some new concepts (namely: the condensation of a many-valued random variable, many-valued Borel-measurable function, and almost-certain-continuity); the definitions and the precise statement of the result are too complicated to be reproduced here.

P. R. Halmos (Chicago, Ill.).

*Gnyegyenko, B. V., és Kolmogorov, A. N. **Független valószínűségi változók összegeinek határeloszlásai.** [Limit distributions for sums of independent random variables.] Akadémiai Kiadó, Budapest, 1951. 256 pp. 32.00 florints.

Translation of Gnedenko and Kolmogorov, Predel'nye raspredeleniya dlya summ nezavisimyh slučaĭnyh veličin [Gostehizdat, Moscow-Leningrad, 1949; these Rev. 12, 839]. The translation is by I. Földes.

Kawata, Tatsuo. On the central limit theorem and the moment generating function. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 3, 6–21 (1951).

Several equivalent theorems are proved for the convergence to $e^{t^2/2}$ of the moment generating functions $E(e^{tX_k})$ of a sequence of independent random variables X_k with zero mean and finite variances satisfying some auxiliary condi-

tions. From these is derived a sufficient condition, resembling Lindeberg's, for the convergence of the moments of X_k to those of a normal distribution. *K. L. Chung.*

Snell, J. L. Applications of martingale system theorems. Trans. Amer. Math. Soc. **73**, 293–312 (1952).

An abstract of this paper has been reviewed [Abstract of a thesis, Univ. of Illinois, 1951; these Rev. **13**, 568]. In the first part of the paper several results are proved extending and amplifying some of Doob's results on semimartingales (to be published in a forthcoming book). They require too many preliminaries to be quoted here, however it should be noted that the extension, as compared with Doob's published version, lies in the following directions: (1) conditional expectations with respect to Borel fields are considered (in fact, a stochastic process is defined to be a collection of random variables and appropriate Borel fields); (2) infinite expectations are allowed (ergo, "generalized"). Doob's fundamental theorem on the number of upcrossings (see the cited review) for a martingale is extended to a generalized semimartingale using ideas which are almost obvious under the gambling interpretation. The second part of the paper applies the semimartingale theory to a problem in sequential decision (game) theory concerning the best stopping rule in the sense of minimizing the loss. Accordingly, terms like "stopping variable", "(ϵ) (minimizing) stopping pair", and "maximal generalized semimartingale relative to a stochastic process" are introduced to formalize some of the concepts in decision theory. This part should be interesting to a reader who prefers a formal treatment to an "intuitive" one. Misprints noted: read $\geqq$ for $\leqq$ in (2.2.1) and Definition 3.4. *K. L. Chung.*

Kolmogorov, A. N. On the differentiability of the transition probabilities in stationary Markov processes with a denumberable number of states. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 53–59 (1951). (Russian)

The author considers Markov chains with infinitely many states and stationary transition probabilities. Let $[p_{ij}(t)]$ be the matrix of transition probabilities for time t. It is assumed that $\lim_{t\to 0} p_{ii}(t)=1$. The reviewer has shown [Trans. Amer. Math. Soc. **52**, 37–64 (1942); these Rev. **4**, 17] that then $p_{ij}'(0)=a_{ij}$ exists for $j\neq i$, and for $j=i$ if $a_{ii}>-\infty$. The author gives new proofs of these facts, proving also that a_{ij} exists and is finite in all cases. He gives simple examples of pathological cases in which, for a single value of i, $a_{ii}=-\infty$, and in which every a_{ij} is finite but, for a single value of i, $\sum_j a_{ij}\neq 0$. In the latter example, the backward differential equations for the transition probabilities are no longer valid. See also the pathological examples given by Lévy [Ann. Sci. Ecole Norm. Sup. (3) **68**, 327–381 (1951); these Rev. **13**, 959]. *J. L. Doob.*

Foster, F. G. On Markov chains with an enumerable infinity of states. Proc. Cambridge Philos. Soc. **48**, 587–591 (1952).

Let the matrix (p_{ij}) satisfy the condition $p_{ij}\geqq 0$, $\sum_{j=1}^{n} p_{ij}=1$ $(i=1, 2, \cdots)$. The solution $\{x_i\}$ of $\sum_j p_{ij}x_j\leqq x_i$ $(i=1, 2, \cdots)$ are called properly divergent (p.d.) if $0\leqq x_i<\infty$ $(i=1, 2, \cdots)$ and $\lim_{i\to\infty} x_i=\infty$. Let π_{ij} be the "time average" $\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} p_{ij}^{(m)}$, $(p_{ij}^{(m)})=(p_{ij})^m$. A result due to Kendall [same Proc. **47**, 633–634 (1951); these Rev. **13**, 51] states that if there exists a p.d. solution, then $\sum_j \pi_{ij}=1$ for all i and there exists a finite set C of states j such that $\sum_{k\in C} p_{ij}=1$ for $i\in C$. The author proves that the converse is true under the following three conditions. (i) For each i non-ϵ C, there exists t such that $\sum_{k\in C} p_{ik}^{(t)}>0$. (ii) For each i and j non-ϵ C $(i<j)$, there exists t such that $p_{ij}^{(t)}>0$. (iii) There exists a number δ, $0\leqq\delta<1$, such that if $t_i(\delta)$ denotes the number of positive integers t for which $\sum_{k\in C} p_{ik}^{(t)}\leqq\delta$, then $\lim_{i\to\infty} t_i=\infty$. *K. Yosida* (Nagoya).

Daniels, H. E. The statistical theory of stiff chains. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 290–311 (1952).

The author investigates Markoff chains in two and three dimensions. The transition probability is relative to lengths and orientations of successive links. Moving axes and characteristic functions are used. Also, limiting cases and asymptotic solutions are discussed. *M. Loève.*

Klein, G. A generalization of the classical random-walk problem, and a simple model of Brownian motion based thereon. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 268–279 (1952).

Comparison of Brownian motion with random walk having persistence, which corresponds to the inertia of the particle. The method of finite differences is used. Cases of initial thermal equilibrium and arbitrary initial kinetic energy are investigated. *M. Loève* (Berkeley, Calif.).

Davis, R. C. On the theory of prediction of nonstationary stochastic processes. J. Appl. Phys. **23**, 1047–1053 (1952).

L'auteur considère deux fonctions aléatoires du second ordre $S(t)$ (signal) et $N(t)$ (bruit) réeles, en général non stationnaires; $E[N(t)]=0$; $E[S(t)]=m(t)$ est inconnue; les covariances:

$$E\{[S(s)-m(s)][S(t)-m(t)]\},$$
$$E\{[S(s)-m(s)][S(t)-m(t)+N(t)]\},$$
$$E\{[S(s)-m(s)+N(s)][S(t)-m(t)+N(t)]\}$$

sont connues, et continues pour $0\leqq s, t\leqq T$, $T>0$ fini; en supposant que $m(t)$ est un polynome de degré n à coefficients inconnus, l'auteur montre que parmi les "prédictions" de $S(T+\Delta)$ $(\Delta>0)$ de la forme

$$S^*(T+\Delta)=\int_0^T K(t)[S(t)+N(t)]dt$$

telles que $E[S^*(T+\Delta)]=m(T+\Delta)$ il y en a une et une seule pour laquelle $E\{[S(T+\Delta)-S^*(T+\Delta)]^2\}$ est minimum [les fonctions aléatoires intervenant sont développées en séries de fonctions orthonormées sur $(0, T)$]. La "meilleure prédiction" peut aussi se définir comme l'espérance mathématique de $S(T+\Delta)$, conditionnelle connaissant $S(t)+N(t)$ pour $0\leqq t\leqq T$; l'auteur examine ce point de vue, qui se confond avec le précedent si $S(t)$ et $N(t)$ sont laplaciens, et qui est particulièrement intéressant si $S(t)$ est de Markoff. *R. Fortet* (Caen).

Dolph, C. L., and Woodbury, M. A. On the relation between Green's functions and covariances of certain stochastic processes and its application to unbiased linear prediction. Trans. Amer. Math. Soc. **72**, 519–550 (1952).

Soient $\{a_k(t)\}$ $(k=0, 1, \cdots, n)$ $n+1$ fonctions de t $(-\infty<t<+\infty)$ de la classe $C^{(n-1)}$, avec $a_0(t)\neq 0$; soit $L_t(y)$ l'opérateur $L_t(y)=\sum_k a_k(t)y^{(n-k)}(t)$ et

$$M_t(y)=\sum_k (-1)^k[a_k(t)y(t)]^{(n-k)}$$

son adjoint; la fonction de Riemann $R(t_0, s)$ correspondant à $L_t(y)=f(s)$ avec: $y^{(k)}(t_0)=0$ $(k\leqq n-1)$ est la solution unique de $M_s[R(t_0, s)]=0$ pour laquelle $\partial^k R(t_0, s)/\partial t_0^k|_{s=t_0}=0$

si $k<n-1$, $=(-1)^{n-1}/a_0(t)$ si $k=n-1$; résultat analogue pour la fonction de Riemann adjointe $R^*(t_0, s)$. Soit $\xi(t)$ un processus de Wiener-Levy; la covariance

$$r(t, s)=\int_{-\infty}^{t} R(t, v)R(s, v)dv$$

du processus $y(t)=\int_{-\infty}^{t} R(t, v)d\xi(v)$, solution de l'équation (formelle) $L_t(y)=\xi'$, a diverses propriétés, en particulier pour $t\neq s$ elle satisfait en t ou en s à l'équation self-adjointe formelle $M_tL_t(y)=0$. Les auteurs considèrent l'équation $f(t)=\int_0^T r(t, s)dw(s)$ où f et r sont données, et en construisent (sous certaines conditions) une solution dans le cas où r est de la forme $r=\phi_1(t)\phi_2(s)$; généralisation et extension à l'équation $f(t)=\int_0^T[z(t, s)+r(t, s)]dw(s)$ où $z(t, s)$ donnée est aussi une covariance. Application au problème suivant: soient (a) $S(t)$ un processus (signal) du second ordre de covariance $z(t, s)$ connue, (b) $N(t)$ un processus (bruit) du second ordre de covariance $r(t, s)$ connue, d'espérance mathématique nulle, non correlé avec $S(t)$; on observe $S(t)+N(t)$ pendant un intervalle de temps fini $(0, T)$; déduire linéairement de cette observation une "prédiction" de $S(\tau)$ $(\tau>T)$ sans déviation et rendant minimum la variance de l'erreur; les auteurs traitent le cas où $E[S(t)]=\sum_{k=0}^{n} a_k g_k(t)$ où les g_k sont des fonctions connues, les a_k étant inconnus; autres problèmes de type voisin et exemples. *R. Fortet* (Caen).

Keilson, Julian, and Storer, James E. On Brownian motion, Boltzmann's equation, and the Fokker-Planck equation. Quart. Appl. Math. **10**, 243–253 (1952).

A much needed comparison between solutions of Boltzmann's integral equation and the corresponding Fokker-Planck equation. A kernel which describes reasonably well the microscopic scattering process is selected. Then the integral equation is solved. It is shown that the Fokker-Planck solution is reliable as far as first and second moments are concerned and still reliable for moments of higher order if the dispersion is relatively large. *M. Loève.*

Ramakrishnan, Alladi. Stochastic processes relating to particles distributed in a continuous infinity of states. Proc. Cambridge Philos. Soc. **46**, 595–602 (1950).

The author considers particles distributed in a continuous infinity of states characterized by a parameter E, the distribution varying with the time t; $M(E, t)$ represents the stochastic variable denoting the number of particles with parametric values less than E, at time t; $dM(E, t)$ is thus the number of particles in the range dE; the fundamental assumptions are:

$$\text{expectation of } dM(E, t)=f_1(E, t)dE$$
$$P(1)=f_1(E, t)dE+O(dE)^2$$
$$P(n)=O(dE)^n, \quad n>1,$$

where $P(n)$ denotes the probability that n particles occur in dE. The main result is an expression of the expectation of $[M(E_a, t)-M(E_b, t)]^r$ in terms of integrals of the "product densities" $f_s(E_1, \cdots, E_s, t)$ which are defined as expectations of $dM(E_1, t)\cdots dM(E_s, t)$. Application is made to the statistical problem of cosmic radiation; similar methods have been developed in problems on population by Kendall [J. Roy. Statist. Soc. Ser. B. **11**, 230–264 (1949); these Rev. **11**, 672] and Bartlett and Kendall [Proc. Cambridge Philos. Soc. **47**, 65–76 (1951); these Rev. **12**, 620]. *J. Kampé de Fériet* (Lille).

Giesekus, Hanswalter. Morphologisch homogene Funktionen und ihre Erzeugung durch statistische Superposition von Elementfunktionen. I. Allgemeine Theorie. Z. Angew. Math. Mech. **30**, 154–168 (1950). (German. English, French, and Russian summaries)

Giesekus, Hanswalter. Morphologisch homogene Funktionen und ihre Erzeugung durch statistische Superposition von Elementfunktionen. II. Einige Hinweise und Beispiele zur Anwendung der Theorie. Z. Angew. Math. Mech. **30**, 215–223 (1950). (German. English, French, and Russian summaries)

The author calls "morphologisch homogene Funktion" any $F(t)$ such that: (1) $F(t)\in L^2$ in every finite interval; (2) $\lim|G(\omega)|^2/(T_2-T_1)$, $T_1\to-\infty$, $T_2\to+\infty$, exists, where $G(\omega)$ is the Fourier transform of the truncated function $(=0$ outside of $T_1<t<T_2)$. The theory developed in Part I is thus a particular case of the generalized harmonic analysis introduced by N. Wiener [Acta Math. **55**, 117–258 (1930)] whose name is not mentioned. In Part II, some physical applications are given, in particular, to the area of the sun-spots. *J. Kampé de Fériet* (Lille).

Arfwedson, G. A semi-convergent series with application to the collective theory of risk. Skand. Aktuarietidskr. **35**, 16–35 (1952).

A continuation of an earlier paper [Skand. Aktuarietidskr. **33**, 1–38 (1950); these Rev. **12**, 191]. It is proved that

$$\frac{2}{\pi^{1/2}}\int_0^\infty \frac{e^{-xz\beta^2}\beta^{2p}d\beta}{(1-\beta^2)^{1/2}}=\sum_{n=0}^{n-1}\frac{k_n}{z}\cdot\frac{(2n+2p-1)!}{(2z^2)^{n+p}}+R_n,$$

$$k_n=(2n-1)!/(2n)!, \quad k_0=1,$$

$$R_n<z^{-2n-2p-1}2^{1/2}(n/e)^n\{1+(\tfrac{1}{2}\pi(n+\tfrac{1}{2}))^{1/2}\}^{2p}$$

provided $R_n>0$; the latter requirement leads to inequalities connecting n, p and z. This series is applied to the determination of the probability distribution of the total losses suffered by an insurance company up to the time when its net risk premium income has amounted to x, its risk loading factor on these premiums being λ, and its initial risk reserve u. It is assumed that the probability of a risk of size z falling in is e^{-z}, independent of time. *H. L. Seal.*

Gomm, Gerhart. Wahrscheinlichkeitsprobleme im Fernsprechverkehr. Mitteilungsblatt Math. Statist. **4**, 183–204 (1952).

This is an expository article on telephone trunking, covering such topics as loss and delay in simple trunk groups, loss in graded multiples, changes in character of traffic after passage through one or more trunks, and the effect of slow variations in call input. *J. Riordan* (New York, N. Y.).

Mathematical Statistics

Tippett, L. H. C. The methods of statistics. 4th ed. John Wiley & Sons, Inc., New York, N. Y.; Williams & Norgate, Ltd., London, 1952. 395 pp. $6.00.

Wishart, John. Moment coefficients of the k-statistics in samples from a finite population. Biometrika **39**, 1–13 (1952).

D'une population de N individus pour lesquels un caractère prend les valeurs $X_1, X_2, \cdots, X_N$, on extrait au

hasard (tirage en grappe) un groupe de n individus pour lesquels le caractère prend les valeurs $x_1, x_2, \cdots, x_n$ (nombres pris aléatoirement parmi les X_j). L'auteur définit après Tukey des fonctions $k_{r,s,\ldots}$ symétriques des x_j; elles forment un système complet de fonctions symétriques (en particulier, les fonctions symétriques élémentaires des x_j s'expriment en fonction des $k_{r,s,\ldots}$) qui comprennent comme cas particulier les statistiques k_r de Fisher. Soit $K_{r,s,\ldots}$ la fonction symétrique des X_j analogue à $k_{r,s,\ldots}$ pour les x_j; si $N=+\infty$, $K_{r,s,\ldots}$ est égal au produit des cumulants d'ordre $r, s, \cdots$; on a d'après Tukey: $E(k_{r,s,\ldots})=K_{r,s,\ldots}$. L'auteur calcule les produits et puissances des k_r en fonctions linéaires des $k_{r,s,\ldots}$ et en déduit leurs espérances mathématiques; il donne des résultats complets jusqu'au 6ième ordre inclus et partiels pour les 7ième et 8ième ordres. *R. Fortet.*

Tintner, Gerhard. Die Anwendung der Variate Difference Methode auf die Probleme der gewogenen Regression und der Multikollinearität. Mitteilungsblatt Math. Statist. **4**, 159–162 (1952).

The author considers a number of observation series whose constituent terms may be represented as the sum of a systematic component and a random residual. The residuals may be arbitrarily correlated but not autocorrelated, and the systematic terms are assumed to obey R linearly independent linear relations. If the estimated covariance matrices of the observations and residuals are A, V respectively (the latter being obtained as the limiting covariance matrix of the repeatedly differenced observations), then R may be estimated from the magnitudes of the roots λ_j of $|A-\lambda V|=0$. The corresponding eigenvectors yield estimates of the linear relations. *P. Whittle* (Uppsala).

Wold, Herman. Über gleichmässige Interkorrelation. Mitteilungsblatt Math. Statist. **4**, 163–166 (1952).

Let there be given n identically distributed random variables x_i, $i=1, 2, \cdots, n$, where any pair (x_i, x_j), $i\neq j$, are correlated with the same intra-class correlation coefficient ρ. The author shows elegantly that by considering the x_i's as observations on a suitably defined stationary process, it is obvious that any statistic of the form $u(x_1-\bar{x}, x_2-\bar{x}, \cdots, x_n-\bar{x})$ must have exactly the same probability distribution function as in the case where the x_i's are mutually independent. The recent results of M. Halperin [Ann. Math. Statistics **22**, 573–580 (1951); these Rev. **13**, 261] fall within the scope of this paper. *B. Epstein* (Detroit, Mich.).

King, Edgar P. The operating characteristic of the control chart for sample means. Ann. Math. Statistics **23**, 384–395 (1952).

Consider m samples of size n each, the samples being $N(\mu_i, \sigma^2)$ while the μ_i are $N(\mu, \theta^2\sigma^2)$. Let $\bar{x}_i$, $\bar{\bar{x}}$, and $\bar{r}$ denote the ith sample mean, the overall mean, and the mean of sample ranges. Let β_0, β be respectively the probability that all the $|\bar{x}_i-\bar{\bar{x}}|$ are less than $3\sigma/\sqrt{n}$, $3\bar{r}\sigma/\sqrt{n}E(\bar{r})$. The author tables β_0 and β to 2D for $\theta=0(0.5)3$, $m=2, 3, 4$, $n=5, 10$, and sometimes 2; and gives bounds on them for some other values. These probabilities occur in the power function of the test that $\theta=0$ used in quality control.
J. L. Hodges, Jr. (Berkeley, Calif.).

Krishna Iyer, P. V. Further contributions to the theory of probability distributions of points on a line. III. J. Indian Soc. Agric. Statistics **4**, 50–71 (1952).

Continuing the program of several previous papers [e.g., same J. **2**, 141–160 (1949); **3**, 80–93 (1950); these Rev. **12**, 271; **13**, 142], the author's purpose is to give simplified calculations of factorial moments of various (generalized) distributions of the number of positive differences between successive observations, runs up and down, peaks and troughs, etc., associated with various nonparametric tests proposed by several authors. A new distribution is considered which has applications to testing nonparametrically whether k samples come from the same population and to randomized block experiments. *J. Kiefer.*

Lukacs, Eugene. The stochastic independence of symmetric and homogeneous linear and quadratic statistics. Ann. Math. Statistics **23**, 442–449 (1952).

The distributions of the mean and of the variance in samples from a continuous population are known to be stochastically independent if and only if the parent distribution is normal, which has been verified by R. C. Geary, T. Kawata and H. Sakamoto, and the present author. In this paper the following more general theorem is proved by solving the differential equation for characteristic functions without assuming probability density functions.

If a univariate distribution has moments of first and second order and admits a homogeneous and symmetric quadratic statistic Q which is distributed independently of the mean of a sample of n drawn from this distribution, then it is either the normal distribution (Q is then proportional to the variance), or the degenerate distribution (in this case no restriction is imposed on Q), or a step-function with two symmetrically located steps (in this case Q is the sum of the squared observations). The converse of this statement is also true. *T. Kitagawa* (Fukuoka).

Mihalevič, V. S. On the mutual disposition of two empirical distribution functions. Doklady Akad. Nauk SSSR (N.S.) **85**, 485–488 (1952). (Russian)

This is a continuation of recent researches of Gnedenko and others sharpening the results of Smirnov on the comparison of two empiric distribution functions [Gnedenko and Korolyuk, same Doklady **80**, 525–528 (1951); Gnedenko and Rvačeva, ibid. **82**, 513–516 (1952); Gnedenko, ibid. **82**, 661–663 (1952); Gnedenko and Mihalevič, ibid. **82**, 841–843; **85**, 25–27 (1952); these Rev. **13**, 570, 760; **14**, 60]. Let $x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}$ be independent observations on a random variable with a continuous distribution and let $F_1(x)$ and $F_2(x)$ be, respectively, the empiric d. f.'s determined by the x's and y's. Put $T_{n_1n_2}(x, z)=F_2(x)+zm^{-1/2}$ where $z\geqq 0$ and $m=(n_1n_2(n_1+n_2)^{-1})^{1/2}$. The point x_k is said to be a crossing point if $F_1(x_k-0)>T_{n_1n_2}(x, z)<F_1(x_k+0)$. Let $v(z, n_1, n_2)$ be the number of crossing points among $x_1, \cdots, x_{n_1}$. The exact distribution of v is given in the case $n_1=n_2=n$, as follows: Writing $c=[z(2n)^{1/2}]$, $k=[l(2n)^{1/2}]$, we have $P\{v(z, n, n)<l(2n)^{1/2}\}=0$ if $k\leqq 0$, $=1-C_{2n}^{n-k-c}/C_{2n}^{n}$ if $0<k$, $c+k\leqq n$, and $=1$ if $c+k>n$. Furthermore, let $C_n(z)$ be the number of x_k, $1\leqq k\leqq n$, for which $F_1(x_k+0)>T_{nn}(x_k, z)$. An exact formula for $P\{C_n(z)<nt\}$ is also given.
K. L. Chung (Ithaca, N. Y.).

Fraser, D. A. S. Sufficient statistics and selection depending on the parameter. Ann. Math. Statistics **23**, 417–425 (1952).

Sufficient statistics are studied at the abstract level illustrated by Halmos and Savage [same Ann. **20**, 225–241 (1949); these Rev. **11**, 42]. Two ideas are introduced and explored. First, a function $T(x)$ is called "f-sufficient" for a set of probability densities $\{f_\alpha(x)\}$ relative to a (σ-finite) measure λ, if $f_\alpha(x)=g_\alpha(T(x))h(x)$, where none of the func-

tions concerned are required to be measurable (except the $f_\mu(x)$). It is shown, for example, that there exists exactly one minimal f-statistic. Second, the "selection statistic" of $\{f_\mu(x)\}$ is by definition the set-valued function $T(x)=\{x' \mid f_\mu(x')\neq 0$ for all μ such that $f_\mu(x)\neq 0\}$. The $\{f_\mu(x)\}$ for which the selection statistic is f-sufficient are characterized. It is shown that the minimal f-sufficient statistic is the selection statistic together with the minimal f-sufficient statistic of a set of densities with a common carrier. The point is made and illustrated that, with good fortune, f-sufficient statistics often turn out to be sufficient statistics. *L. J. Savage* (Chicago, Ill.).

Lehmann, E. L. On the existence of least favorable distributions. Ann. Math. Statistics **23**, 408–416 (1952).

A "least favorable" distribution is one such that, if the possible states to be presented to the statistician were chosen at random (from a space Ω) in accordance with this distribution, the minimum loss of the statistician would be maximized. For many problems one can, by Helly's theorem, choose a convergent sequence of distribution functions on the space Ω such that the minimum loss according to the members of this sequence approaches the supremum (with respect to all a priori distributions) of all minimum Bayes losses. The limit of these distribution functions need not be a distribution function because it may have variation less than one (i.e., some probability has "moved out to infinity"). When this does not occur, the limiting distribution function is, under general conditions, a least favorable a priori distribution. For several important statistical problems the author gives sufficient conditions to ensure that there be no such movement of probability to infinity. The conditions do not require the compactness of Ω, and do not lend themselves to convenient summary. *J. Wolfowitz.*

Godambe, V. P. On two-stage sampling. J. Roy. Statist. Soc. Ser. B. **13**, 216–218 (1951).

Consider a two-stage sampling plan in which we decide, in advance of the first stage, for each primary unit, how many observations will be made there if the unit is selected. The total number m of observations becomes a random variable, but it is possible in this way to obtain estimates with smaller variance than can be obtained if the sample size is fixed at $E(m)$. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Peterson, Raymond P. Constant risk minimax point estimates. Univ. Washington Publ. Math. **3**, 77–84 (1952).

For the problem of estimation of a parameter $\theta \in \Omega \subset E^n$, the author proves for the absolutely continuous case with fixed sample size such well-known results as the one that a constant risk (CR) Bayes estimator is minimax; that a CR minimax estimator is uniformly best (UB) among the CR estimators; and that (*) if the density function $p(x, \theta)$ and weight function $W(\delta, \theta)\geqq 0$ depend only on $(x-\theta)$ and $(\delta-\theta)$ respectively and $\delta^0(x_1, \cdots, x_n)+c=\delta^0(x_1+c, \cdots, x_n+c)$, then the risk function $r(\theta, \delta^0)$ of δ^0 is constant (similarly for scale, scale and location, etc.). Also that if $r(\theta, \delta^0)<\infty$ and $\varphi_{\delta^0}(x)=\int_\Omega W(\delta^0(x), \theta)p(x, \theta)d\theta<\infty$ and $\delta^0(x)$ minimizes $\varphi_\delta(x)$ and δ^0 is CR, then δ^0 is minimax and UBCR, with an obvious application to (*) (etc.). [See, e.g., Wald, Statistical decision functions, Wiley, New York; Ann. Math. Statistics **10**, 299–326 (1939), especially Theorem 5; these Rev. **12**, 193; **1**, 152; Peisakoff, Thesis, Princeton Univ., 1951; Kallianpur, abstract in Ann. Math. Statistics **21**, 310–311 (1950).] *J. Kiefer* (Ithaca, N. Y.).

Peterson, Raymond P. The determination of classes of constant risk estimates. Univ. Washington Publ. Math. **3**, 85–87 (1952).

The main result seems to be that, for the problem of the preceding review, if δ is a decision function for which $W(\theta, \delta(x_1, \cdots, x_n))$ is a function only of quantities $t_i=\eta(x_i, \theta)$ $(i=1, \cdots, n)$ where the joint distribution of the t_i's does not depend on θ, then the expected value of W is independent of θ. Examples are given. *J. Kiefer* (Ithaca, N. Y.).

Lal, D. N. On the test of a hypothesis concerning two independent frequency distributions. J. Indian Soc. Agric. Statistics **4**, 72–84 (1952).

The author considers the problem of testing whether or not two distribution functions are the same, where it is known that both are members of a family depending on s parameters and the test is based on random samples which are grouped into $k\geqq s+1$ divisions. When the sample sizes are large, the distribution of an appropriate quadratic form in the difference of the vectors of maximum likelihood estimators from the two samples tends to the χ^2 distribution with s degrees of freedom if the populations are the same; the likelihood ratio test gives a similar result. The method is familiar and the result is well known [see, e.g., S. S. Wilks, Ann. Math. Statistics **9**, 60–62 (1938); A. Wald, Trans. Amer. Math. Soc. **54**, 426–482 (1943); these Rev. **7**, 20]. *J. Kiefer* (Ithaca, N. Y.).

Darling, D. A. On a test for homogeneity and extreme values. Ann. Math. Statistics **23**, 450–456 (1952); correction, **24**, 135–136 (1953).

$X_1, X_2, \cdots, X_n$ étant des variables aléatoires positives indépendantes de même fonction de répartition $F(x)$ de densité $\phi(x)$, soit $Z_n=\sum_{j=1}^n X_j/\max(X_1, \cdots, X_n)$; la caractéristique $\phi(t)$ de Z_n est

$$\phi(t)=ne^{it}\int_0^{+\infty}\left(\beta\int_0^1 e^{it\alpha}\phi(\alpha\beta)d\alpha\right)^{n-1}\phi(\beta)d\beta;$$

on en déduit des formules donnant les moments de Z_n, par exemple:

$$E(Z_n)=1+n(n-1)\int_0^{+\infty}[F(\beta)]^{n-2}\times\left\{\int_0^1\phi(\beta u)du-\phi(\beta)\right\}\phi(\beta)d\beta;$$

dans certains cas [si les X_j sont uniformes sur $(0, 1)$, si $\phi(x)=e^{-x}x^{k-1}/\Gamma(k)$ avec k entier >0], on peut en déduire la fonction de répartition de Z_n. L'auteur obtient la distribution-limite de Z_n pour $n\to+\infty$ sous des conditions assez générales. Ces résultats peuvent être utilisés dans des tests d'homogénéité. *R. Fortet* (Caen).

Walsh, John E. Some bounded significance level properties of the equal-tail sign test. Ann. Math. Statistics **22**, 408–417 (1951).

Let $x_1, \cdots, x_n$ be independent random variables and let $y_1<\cdots<y_n$ be the same ordered according to size. Bounds on $P[y_i<0$ or $y_{n+1-i}>0]$ are tabled as a function of $\beta=\max_j \max[|P(x_j<0)-\frac{1}{2}|, |P(x_j>0)-\frac{1}{2}|]$, for $\beta=.02$, .08, .1, .15, and .2, and for 21 pairs (n, i). *J. L. Hodges, Jr.* (Berkeley, Calif.).

Anderson, T. W., and Darling, D. A. Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. Ann. Math. Statistics **23**, 193–212 (1952).

Soient X_j $(j=1, 2, \cdots, n)$ n variables aléatoires indépendantes de même fonction de répartition continue $F(x)$;

$F_n(x)$ la fonction de répartition empirique déduite de l'observation des X_j; $\psi(t)$ $(0 \leqq t \leqq 1)$ une fonction $\geqq 0$ quelconque (poids). Les auteurs cherchent à déterminer les fonctions de répartition asymptotiques, pour $n \to +\infty$, des variables aléatoires W_n et K_n définies par

$$W_n^2 = n \int_{-\infty}^{+\infty} [F_n(x) - F(x)]^2 \psi[F(x)]\, dF(x),$$

$$K_n = \sup_x \{n^{1/2}[F_n(x) - F(x)](\psi[F(x)])^{1/2}\}.$$

On sait que divers auteurs (v. Mises, Cramér, Kolmogoroff, Smirnov) ont etudié le cas particulier $\psi(t) \equiv 1$. Les auteurs, suivant une idée de Doob, introduisent systématiquement un processus laplacien limite, et démontrent que: (a) si $\psi(t)$ est continue, l'équation $\partial^2 h/\partial t^2 + \lambda\psi(t)h = 0$ a pour $\lambda \geqq 0$ une solution $h(t, \lambda)$ et une seule telle que $h(0, \lambda) = 0$, $\partial h/\partial t|_{t=0} = 1$, et la caractéristique de la fonction de répartition limite de W_n^2 est $[h(1, 0)/h(1, 2it)]^{1/2}$; (b) pour K_n, le problème se ramène sous des conditions très générales, à la recherche de probabilités d'absorption pour un processus laplacien de Markoff, donc à la résolution d'équations aux dérivées partielles paraboliques. *R. Fortet* (Caen).

Walker, A. M. Some properties of the asymptotic power functions of goodness-of-fit tests for linear autoregressive schemes. J. Roy. Statist. Soc. Ser. B. **14**, 117–134 (1952).

The author considers the tests of fit for a linear autoregressive scheme developed by Quenouille (the Q test) and Bartlett and Diananda (the BD test). The asymptotic power function is evaluated for certain immediately neighbouring hypotheses. As a specimen of the results obtained, it is shown that, while the relative powers of the Q and BD tests depend upon the particular choice of counterhypothesis, a certain modification of the Q test is at least as powerful as either over a certain class of counterhypotheses. The likelihood ratio L for two autoregressive schemes is also considered, and the author shows that on the hypothesis of the lesser order autoregression $-2 \log L$ is asymptotically distributed as a χ^2 variate. Finally, it is shown that the modified Q test is asymptotically equivalent to the likelihood test in the sense that the two test statistics differ by a term of relative order $n^{-1/2}$ if the parameters of the counterhypothesis differ by not more than $O(n^{-1/2})$ from those of the null hypothesis. *P. Whittle* (Uppsala).

Hodges, J. L., Jr., and Lehmann, E. L. The use of previous experience in reaching statistical decisions. Ann. Math. Statistics **23**, 396–407 (1952).

As a possible method of using past experience in selecting a decision procedure (dp), the authors suggest a compromise between the Bayes and minimax approaches; namely, to use a restricted Bayes solution (rbs) δ_0 which minimizes $R_\delta^*(\lambda) = \int R_\delta(\theta)\, d\lambda(\theta)$ subject to $R_\delta(\theta) \leqq C_0$, where $R_\delta(\theta)$ is the risk function of δ, λ is an a priori distribution, and $C_0 > \inf_\delta \sup_\theta R_\delta(\theta)$. This is equivalent to using a δ_0 which minimizes $\rho R_\delta^*(\lambda) + (1-\rho) \sup_\theta R_\delta(\theta)$ for some ρ with $0 \leqq \rho \leqq 1$. The authors obtain sufficient conditions for a dp to be a rbs and for the existence of a rbs. They also show under weak regularity conditions that if λ is an a priori distribution and $r_\lambda(C)$ is the restricted Bayes risk corresponding to $C_0 = C$, then $r_\lambda(C)$ is convex, continuous, and strictly decreasing, and the limit of a sequence of rbs's (in the sense of Wald's regular convergence) is a rbs. Extensions and examples are given. *J. Kiefer.*

Paulson, Edward. On the comparison of several experimental categories with a control. Ann. Math. Statistics **23**, 239–246 (1952).

Certain multi-decision problems arising in the determination of the "best" of k categories when comparing $k-1$ experimental categories with a standard or control are investigated. Given a sample consisting of kn independent observations $\{x_{ij}\}$ $(i = 1, 2, \cdots, k;\ j = 1, 2, \cdots, n)$ where x_{ij} is the jth observation with category Π_i, one problem is to devise a procedure for selecting one out of the k categories as best so that if none of the experimental categories $\Pi_2, \cdots, \Pi_k$ is actually superior to Π_1, then the probability that Π_1 is selected will be $\geqq 1-\alpha$. The author considers this and related problems and results of both an exact and approximate nature are obtained for the cases where (a) the observations with each category are normally distributed and (b) when the observations with each category have a binomial distribution. *R. P. Peterson* (Seattle, Wash.).

Wolfowitz, J. On the stochastic approximation method of Robbins and Monro. Ann. Math. Statistics **23**, 457–461 (1952).

The author gives weaker conditions for the convergence of the Robbins-Monro stochastic approximation method than those given in the paper of Robbins and Monro [same Ann. **22**, 400–407 (1951); these Rev. **13**, 144].

R. P. Peterson (Seattle, Wash.).

Kiefer, J., and Wolfowitz, J. Stochastic estimation of the maximum of a regression function. Ann. Math. Statistics **23**, 462–466 (1952). See the corresponding note p. 1278

The authors define a recursive stochastic approximation scheme similar to that used by Robbins and Monro [same Ann. **22**, 400–407 (1951); these Rev. **13**, 144] which is shown to converge under somewhat weaker conditions than those given by Robbins and Monro in their paper.

R. P. Peterson (Seattle, Wash.).

Williams, E. J. Some exact tests in multivariate analysis. Biometrika **39**, 17–31 (1952).

Given samples drawn from $q+1$ p-variate normal populations, the variates can be transformed to the new set given by the p sample discriminant functions $x_1, x_2, \cdots, x_p$. The paper gives a method of testing the concordance of the data with a hypothetical discriminant y. The null hypothesis then specifies: (i) the population means of the groups (when represented in a p-dimensional Euclidean hyperspace) fall on a straight line, i.e., the differences between groups are represented by a single parameter; and (ii) the p-dimensional vector y specifies the direction of this line. When $p \leqq q$, the author utilizes a result noted by Bartlett [Ann. Math. Statistics **18**, 1–17 (1947); these Rev. **8**, 474] that the sample discriminant ratio η is a sufficient statistic for the corresponding population parameter, and derives the conditional distribution (given η) of the sample latent roots $\theta_1, \theta_2, \cdots, \theta_p$. This distribution is independent of the unknown parameter and is used in obtaining tests of significance. When $p > q$, $p-q$ of the latent roots are identically zero and the corresponding distribution involves only the remaining latent roots $\theta_1, \theta_2, \cdots, \theta_q$. The cases (i) $p = 2$, $q \geqq p$ and (ii) $q = 2$, $p > q$ are considered in detail, and in each case separate test criteria are also indicated for (a) direction and (b) linearity. A numerical example with $p = 2$ and $q = 7$ is discussed. In the more complex situation when $p \geqq 3$, $q \geqq 3$ the difficulty of using this or any other similar method is pointed out.

R. C. Bose (Chapel Hill, N. C.).

Williams, E. J. The interpretation of interactions in factorial experiments. Biometrika **39**, 65–81 (1952).

In a factorial experiment where the joint effects of two or more factors are not additive, the author proposes a model where the effects of one factor are assumed to be proportional (rather than equal as in the conventional model) at different levels of the other factors. When there are two factors A and B, the expected effect of the combination of level i of A with level j of B may be expressed in the form $a_ic_j+b_j$. The main effects for the different levels of A would then be represented by the constants a_i which are determinate apart from a constant factor, the constants c_j being regarded as weights to be applied to the results at different levels of B, in determining the A effects. The effects of B are given by the constants b_j. If x_{ij} is the observed mean for level i of A in combination with level j of B, the author takes the equations of expectation in the normalized form $E(x_{ij})=a_ic_j\theta+b_j$, $\sum a_i=\sum b_j=0$, $\sum a_i^2=\sum c_j^2=1$, where θ is a constant. It is shown that weights are estimated by the latent vector of a matrix T of sums of squares and products corresponding to the largest latent root of the matrix, and that the sum of squares for the weighted main effect is a multiple of this root. The other latent roots correspond to a partition of the interaction sum of squares. The null hypothesis that with main effects suitably defined, interactions do not exist, is equivalent to assuming that the matrix ϕ of the expected effects of factor A is of rank 1, that is, the matrix $\phi=r\phi'\phi$ has only one non-zero root λ, r being the number of replications. A test for this null hypothesis is given for large λ. A test for the adequacy of a set of proposed weights is also given under the same assumption, together with an exact test when T has only two non-vanishing latent roots. The analysis is illustrated by a numerical example.

R. C. Bose (Chapel Hill, N. C.).

Robbins, Herbert. Some aspects of the sequential design of experiments. Bull. Amer. Math. Soc. **58**, 527–535 (1952).

The author discusses some rarely mentioned but nevertheless very important problems of sequential procedures. Several situations are discussed where two or more populations are involved and at each step the "experimenter" is at liberty to choose the population from which the next item is to be drawn. Moreover the purpose of the sampling is not to test a hypothesis or to estimate a parameter but rather to maximize a given function of the observations. In fact no action may be planned or inference drawn after completion of the sampling process. In this sense the title is somewhat misleading since the paper deals with a sequential aspect of inductive behavior rather than with experimentation. The third problem is the problem of optional stopping. Every hypothesis could be rejected if the experimenter is permitted to discontinue the sampling process whenever it suits him. This situation has been discussed by Feller [J. Parapsychology **4**, 271–298 (1940); these Rev. **3**, 11] as a possible source of bias of experiments on extrasensory perception. *H. B. Mann* (Columbus, Ohio).

Pearce, S. C. Some new designs of Latin square type. J. Roy. Statist. Soc. Ser. B. **14**, 101–106 (1952).

The author remarks first that the inflexibility of the Latin square makes it inapplicable in a great many experimental situations where it would otherwise be very desirable. Its desirable properties are at least partly preserved in certain designs which are derived from Latin squares. The author discusses three designs of this type: Latin squares with a) a column added, b) a column and a row added, c) a column added and a row omitted. Formulae for the analysis of variance of these three designs are given and their efficiency factors are computed. A numerical example for a Latin square with a column added is computed. *H. B. Mann.*

Mathematical Economics

***McKinsey, J. C. C. Introduction to the theory of games.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1952. x+371 pp. $6.50.

This book is intended by the author to be used "as a college textbook for a course for upper division and graduate students". It will certainly find many other readers, however, for it represents the clearest and most direct introduction to game theory currently in print. As such it should be of interest to mathematicians, economists, and any persons curious to learn what game theory is about. The book presupposes a knowledge of calculus, but many portions can be understood without this. Virtually all the material in the book comes from the results of work done during the past decade, including some results as yet unpublished elsewhere. It is extremely comprehensive, treating parts of all the major branches of investigation in the subject, and the reader who absorbs its contents will possess a good over-all picture of the problems, results, and applications of the theory to date.

The first four chapters of the book are concerned with the finite two-person zero-sum game in normal or rectangular form. The concepts are introduced in the first chapter with numerous examples. The fundamental theorem that all such games possess solutions is proved in Chapter 2. Chapter 3 deals with properties of these solutions and gives a method for determining all solutions, while Chapter 4 treats an approximation method for solving a game. Chapters 5 and 6 take up finite games in extensive form. Chapter 5 introduces the notions and shows how to obtain the rectangular form from the extensive form, while Chapter 6 gives the precise definitions and proves the existence of equilibrium points for games with perfect information. The next seven chapters are concerned with infinite two-person zero-sum games. Chapter 7 introduces the necessary definitions. Chapters 8 and 9 on distribution functions and Stieltjes integrals present the mathematical tools necessary for the handling of continuous games which are treated in Chapter 10. Here it is proved that a game on the unit square with continuous pay-off has a value and optimal strategies in distribution functions. Chapter 11 treats the case of separable games, i.e., games whose pay-off function $F(x, y)$ can be written in the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ where f_i and g_i are continuous functions on $[0, 1]$. For these games it is shown that optimal strategies consisting of step functions always exist. Chapter 12 treats convex games, i.e., games whose pay-off function $F(x, y)$ is convex in y for each x. It is shown that the second player always has a pure strategy which is optimal and the first player has step function optimal strategies under suitable conditions. Chapters 13 and 14 are concerned respectively with applying game theory to statistical problems and the relationship between games (finite) and linear programming. Chapters 15, 16, and 17 present the von Neumann theory of n-person games, and the final chapter, 18, mentions some open problems in the theory.

Each chapter concludes with a short section of bibliographical and historical remarks and a set of problems of

varying degrees of difficulty with a strong emphasis on computing solutions of actual games. This is in keeping with the general tone of the book which, as a text rather than a theoretical exposition, lays stress on practical aspects of the theory. It seems a pity, however, that the necessary theorems on convex sets were not proved so that the treatment at least up to the proof of the fundamental theorem in Chapter 2 could be self-contained. The proofs might have been included, for instance, in an appendix.

The book is written with commendable care and precision. Nevertheless, the exposition proceeds in a lucid and amiable fashion. The author has rendered a real service in making accessable to the qualified student a subject of considerable current interest. *D. Gale* (Providence, R. I.).

Kneser, Hellmuth. Sur un théorème fondamental de la théorie des jeux. C. R. Acad. Sci. Paris **234**, 2418–2420 (1952).

The author proves the following generalization of the fundamental theorem of von Neumann for finite zero-sum two-person games [Theory of games and economic behavior, 2d ed., Princeton Univ. Press, 1947; these Rev. **9**, 50]. If (1) K and L are convex spaces over the real numbers (e.g., convex subsets of real vector spaces), (2) $f(x, y)$ is linear in x for each $y \in L$ and linear in y for each $x \in K$, (3) K is compact in a topology for which each function $f(x, y)$ with $y \in L$ fixed is upper semi-continuous, then $\sup_x \inf_y f(x, y) = \inf_y \max_x f(x, y)$. The theorem is established through a "theorem of the alternative" [cf. von Neumann and Morgenstern, loc. cit.] based on the lemma: Let f and g be linear, upper semi-continuous functions on the compact, convex space K such that $\min(f(x), g(x)) < 0$ for all $x \in K$. Then there is a convex combination of f and g which is negative throughout K. It should be remarked that these statements remain true without significant changes in the proof if (2) is replaced by (2′) $f(x, y)$ is concave in x for each $y \in L$ and convex in y for each $x \in K$ and if the lemma is changed to read: Let f and g be concave, upper semi-continuous functions, etc. *H. W. Kuhn.*

Dvoretzky, A., Kiefer, J., and Wolfowitz, J. The inventory problem. II. Case of unknown distributions of demand. Econometrica **20**, 450–466 (1952).

The inventory problem is a particular realization of the more general problem of optimal behavior with respect to data produced by a stochastic process some of whose parameters are subject to control. The first part of this paper [Econometrica **20**, 187–222 (1952); these Rev. **13**, 856] dealt with the case where the probability distribution of demand is completely known. In this part the distribution is known to be only one of a certain class of distributions. The methods used are direct extensions of those of Wald's decision theory, with special emphasis on the calculation of ϵ-complete classes, ϵ-Bayes solution, ϵ-minimax ordering policies [Wolfowitz, Ann. Math. Statist. **22**, 461–465 (1951); these Rev. **13**, 143]. The cases of optimization over one, finitely many, and infinitely many time intervals are treated in turn, with the goal in each case being an ϵ-complete class of ordering policies. Usually ϵ-Bayes solutions are used to sweep out this class. In the case of infinitely many time intervals the stationary situation (distributions and loss functions independent of time) is of special interest. An integral equation is deduced and a convergent iterative method is given for approximating the unique bounded solution which is in turn used to define a class of ordering policies which is ϵ-essentially complete for every positive ϵ. *R. Solow* (Cambridge, Mass.).

Debreu, Gerard. A social equilibrium existence theorem. Proc. Nat. Acad. Sci. U. S. A. **38**, 886–893 (1952).

The author extends the concept of an equilibrium point [Nash, same Proc. **36**, 48–49 (1950); these Rev. **11**, 192] to the following situation: Let $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_\nu$ be the product of the sets $\mathfrak{A}_\iota$ and, for each $a = (a_1, \cdots, a_\nu) \in \mathfrak{A}$ and each $\iota = 1, \cdots, \nu$, let $\bar{a}_\iota$ denote

$$(a_1, \cdots, a_{\iota-1}, a_{\iota+1}, \cdots, a_\nu) \in \mathfrak{A}_\iota = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_{\iota-1} \times \mathfrak{A}_{\iota+1} \times \cdots \times \mathfrak{A}_\nu.$$

Further, for $\iota = 1, \cdots, \nu$, let $A_\iota : \mathfrak{A}_\iota \to \mathfrak{A}_\iota$ be a multivalued function with graph G_ι and let f_ι be a function from G_ι to the completed real line. Finally, $\varphi_\iota(\bar{a}_\iota) = \max_{a_\iota \in A_\iota(\bar{a}_\iota)} f(\bar{a}_\iota, a_\iota)$. Then $a^* \in \mathfrak{A}$ is an equilibrium point if $a_\iota^* \in A_\iota(\bar{a}_\iota^*)$ and $f_\iota(\bar{a}_\iota^*, a_\iota^*) = \varphi_\iota(\bar{a}_\iota^*)$ for $\iota = 1, \cdots, \nu$. Theorem: If, for $\iota = 1, \cdots, \nu$, the set $\mathfrak{A}_\iota$ is a contractible polyhedron, the graph G_ι is closed, the functions f_ι and φ_ι are continuous, and the sets $Ma_\iota = \{a_\iota \in A_\iota(\bar{a}_\iota) \mid f_\iota(\bar{a}_\iota, a_\iota) = \varphi_\iota(\bar{a}_\iota)\}$ are contractible for every $\bar{a}_\iota$, then there exists an equilibrium point. The proof is based on the fixed point theorem of Eilenberg and Montgomery [Amer. J. Math. **68**, 214–222 (1946); these Rev. **8**, 51]. A short historical account is given of the connections between equilibrium points, saddle points, and fixed point theorems for multivalued functions. *H. W. Kuhn.*

Tinbergen, J. Four alternative policies to restore balance of payments equilibrium. Econometrica **20**, 372–390 (1952).

Given m countries trading with and only with each other and initially not all in equilibrium, the author considers the welfare effects of four alternative methods of restoring balance-of-payments equilibrium. The four methods are: discriminatory duties and subsidies; discriminatory duties only; nondiscriminatory duties; devaluation. In each case an optimal representative of each type of policy is selected, the criterion for optimality being a world welfare indicator which is the sum of separate national welfare indicators, each of which is in turn essentially the sum of the areas under the separate Marshallian demand curves. In addition to this special choice, the analysis is conducted under the following strict assumptions. Each country produces one commodity, and uses no imported raw materials in so doing. Each country's output is held constant. The demand in country j for the product of country i is of the form:

$$X^{ij} = \xi^{ij} y^j - \epsilon p^{ij} + \frac{\epsilon}{m-1} \sum_{h \neq i} p^{hj},$$

where the y's are incomes, the p's prices. Thus if each price increases by the same amount, demand remains unchanged. All goods are thus sold under essentially the same demand conditions. The author says that this symmetry is not "of immediate significance."

In this context the author studies for each policy the effect on the volume of trade and the distribution of the (consumption) burden between countries. Among the results are: that only the first and fourth of the above policies lead to no loss in trade; that the second causes smaller loss of trade than the third; that the burden is borne least by the countries with the largest initial balance of payments surpluses; that the burden is most unevenly distributed by

devaluation; and others. See Fleming [Economic J. 61, 48–71 (1951)]. *R. Solow* (Cambridge, Mass.).

Simon, Herbert A. On the application of servomechanism theory in the study of production control. Econometrica 20, 247–268 (1952).

The essence of a system involving a servomechanism is that the output of the system is compared with a desired output and the rate of operation of the system is adjusted to compensate for the error or discrepancy. The formal structure of servomechanisms was first studied in connection with electronics and other engineering applications but applies to many business and economic problems. In particular the operations of a business firm are of this nature when they are guided by a definite inventory policy.

The simplest case is where the firm attempts to control its rate of production so as to keep inventories at a constant level, although it exercises no direct control over withdrawals from inventory, i.e., sales. This problem can be solved by standard methods, using the Laplace transform. It turns out that a sensible procedure would be to adjust the rate of production in accord with a linear function of the "error" (desired inventory minus actual inventory) and the rate of change in the error. The tendency of the system, so controlled, to "hunt" is considered carefully. If the actual change in the rate of production lags appreciably behind the decision to make a change, the system becomes more complicated but still solvable. By these methods a policy can be found which minimizes the costs of production insofar as these costs reflect changes in inventory levels and fluctuations in the rate of production. Thus "servomechanism theory can indeed be applied fruitfully to the analysis and design of decisional procedures for controlling the rate of manufacturing activity." *R. Dorfman* (Berkeley, Calif.).

Boiteux, Marcel. Le "revenu distribuable" et les pertes économiques. Econometrica 19, 112–133 (1951).

All consumers are assumed to maximize their satisfactions according to the same price vector p, all producers their profits according to the same price vector π. If p and π are not collinear, the economy is not in a Pareto optimal state and a loss is incurred for which the following Hicks definition is adopted. An initial optimal state is selected as reference; a neighboring final non-optimal state is considered. Let for the kth consumer, $\delta\rho^k$ be that variation of his income in the initial state which would bring his satisfaction to the value it has in the final state. $-\delta\rho=-\sum_k\delta\rho^k$ is the "revenu distribuable" taken as an expression of the loss associated with the final state for which an approximation formula is given:

$$-\tfrac{1}{2}dp'Xdp+\tfrac{1}{2}d\pi'\Xi d\pi$$

where X (resp. Ξ) is the classical semidefinite negative (resp. positive) aggregate consumption (resp. production) substitution matrix, and dp and $d\pi$ are the variations of p and π from the initial to the final state.

A study, whose exact meaning and mathematical correctness are in doubt for the reviewer, is then made of the following problem. The technology of the firms may be such that the marginal cost pricing policy ($p=\pi$) leads them to have, on the whole, a deficit S. In order to have a smaller given deficit $S-\delta S$ one allows p to differ from π. Let $t=p-\pi$; how should t be chosen, for a given income distribution, so as to minimize the loss resulting from this policy?

G. Debreu (Chicago, Ill.).

Solow, Robert. On the structure of linear models. Econometrica 20, 29–46 (1952).

The author considers the properties of economic models, such as the Leontief input-output model or the Metzler-Goodwin-Chipman multi-sector trade models, which are either of the static form $(I-a)x=c$ or of the dynamic form $Ix(t+1)-ax(t)=c$, where x and c are vectors, I the unit matrix and a a matrix all of whose components are non-negative. In the Leontief model, a is the input-output matrix, c the vector of final demands, and x the vector of total outputs; in the multi-sector models, a_{ij} is the marginal propensity of sector j to buy from sector i, x is the vector of revenues of the sectors, and c the vector of autonomous expenditures.

It is first noted that it follows directly from theorems of Metzler [Econometrica 18, 329–354 (1950); these Rev. 13, 368] and of Hawkins and Simon [ibid. 17, 245–248 (1949), p. 248] that if a is non-negative, then the static system has a non-negative solution for all non-negative c (i.e., $(I-a)^{-1}$ is non-negative) if and only if the corresponding dynamic system is stable, in the sense that all the characteristic roots are less than one in absolute value. Next sufficient conditions for stability are studied. A set of indices (between 1 and n) is said to form a closed set if $a_{pq}=0$ for q in the set, p not. The matrix a is said to be indecomposable if there is no closed set other than the set of all indices. Economically, this implies that an expenditure in one sector will eventually give rise to an expenditure in any other. An indecomposable matrix is said to be cyclic if there is a partitioning of the indices into w mutually disjoint sets $G_1, \cdots, G_w$, such that, for p in G_i, $a_{pq}\neq 0$ only for q in G_{i+1} (here $G_{w+1}=G_1$). Otherwise the matrix is acyclic. Then it is shown that if a is a non-negative indecomposable matrix with no column sum greater than one and at least one less than one, the characteristic roots of a are all less than one in absolute value. Further, under the same hypotheses, the elements of $(I-a)^{-1}$ are not merely non-negative but positive.

The dominant motions of the dynamic system are also studied; it is shown that for a non-negative indecomposable matrix, the dominant motion is monotonic if and only if it is acyclic. In particular, the dominant motion is monotonic if $a_{pp}>0$ for at least one value of p.

Certain extensions to decomposable matrices are also given, as well as the relation of the results to an algebraic theory of Minkowski's. *K. J. Arrow.*

Herstein, I. N. Comments on Solow's "Structure of linear models". Econometrica 20, 685–686 (1952).

The author notes that several results found by Solow in the cited article [see the preceding review] can be deduced very rapidly from some theorems of Wielandt [Math. Z. 52, 642–648 (1950); these Rev. 11, 710] or from the following easy consequence of those theorems: If A is a non-negative indecomposable matrix and $r(A)$ the largest positive characteristic root of A, then $r(A)$ is a strictly increasing function of each element of A.

K. J. Arrow (Stanford, Calif.).

TOPOLOGY

Moise, Edwin E. A remark on $\mathfrak{L}^*$-spaces. Michigan Math. J. 1, 79–80 (1952).

Il existe deux façons de définir une topologie dans le produit cartésien $X \times Y$ de deux espaces $\mathfrak{L}^*$ (au sens de Kuratowski [Topologie I, 1re éd., Warszawa-Lwów, 1933, pp. 76–77]): (I) On peut d'abord définir les topologies dans X et Y de la manière usuelle; puis définir la topologie dans $X \times Y$ de la façon habituelle à partir des topologies définies dans X et Y. (II) On peut dire que $\lim_{i \to \infty}(x_i, y_i) = (x, y)$ si et seulement si $\lim_{i \to \infty} x_i = x$ et $\lim_{i \to \infty} y_i = y$; puis, les notions de convergence et de limite étant aussi définies dans $X \times Y$, en déduire de la manière usuelle une topologie dans $X \times Y$. L'auteur donne un exemple de deux espaces $\mathfrak{L}^*$ qui sont en même temps des espaces de Hausdorff, dont le produit cartésien au sens (I) n'est pas un espace $\mathfrak{L}^*$, et dont le produit cartésien au sens (II) est une espace $\mathfrak{L}^*$ qui est en même temps un espace de Hausdorff discret. *A. Appert.*

Misonou, Yosinao, and Takeda, Ziro. On the compactification of topological spaces. Kōdai Math. Sem. Rep. 1952, 17–18 (1952).

An alternate proof is given of the following well-known results [cf., e.g., P. Alexandroff, Mat. Sbornik N.S. 5(47), 403–423 (1939); these Rev. 1, 318]. For any completely regular S, there exists a continuous mapping of ωS onto βS preserving all points of S; ωS and βS coincide if and only if S is normal (ωS and βS denote, respectively, the Wallman and Čech compactifications). *M. Katětov* (Prague).

Mamuzić, Zlatko. On sets closed or open relative to a set in Hausdorff's space. Bull. Soc. Math. Phys. Serbie 4, nos. 1–2, 25–35 (1952). (Serbo-Croatian. English summary)

If H is a Hausdorff space and S a subset of H, then a set E is said to be closed in S if and only if $E \subset S$ and $E' \cap S \subset E$; likewise for sets open in S. The author generalizes this classical terminology by defining E to be closed relative to S if and only if $E' \cap S \subset E$ (but not necessarily $E \subset S$); similarly for sets open relative to S. The author obtains a number of results relating these concepts to each other and to the classical concepts. *E. E. Floyd.*

Proskuryakov, I. V. On the theory of dimension of topological spaces. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 219–223 (1951). (Russian)

Let R be a topological space not necessarily satisfying any separation axiom but such that disjoint sets A and B of R can be separated by open sets if A and B are closed or if A is closed and B consists of a single point. The Menger-Uryson dimension of R is denoted by $\operatorname{ind} R$, the combinatorial dimension of R by $\dim R$. The author's definition of $\dim R$ is cast in a form different from the usual one: $\dim R \leqq r$ if every finite open covering of R admits a closed finite refinement of multiplicity $\leqq r+1$. This is clearly equivalent to the usual definition in view of the weak normality axiom assumed for R. A simple proof is given of the fact that $\operatorname{ind} R \leqq \dim R$. The proof is based on the following assertion. A sequence $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of families of subsets of R is said to be locally arbitrarily fine if for every $x \in R$ and every neighborhood $U(x)$, there exists an integer N such that $n > N$, $B \in \mathfrak{A}_n$ and $x \in B$ implies that $B \subset U(x)$. Consider a sequence $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ of finite closed coverings of R such that $\mathfrak{F}_{n+1}$ is a subdivision of $\mathfrak{F}_n$ ($n = 1, 2, 3, \cdots$). If such an $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ exists which is locally arbitrarily fine and if every $\mathfrak{F}_n$ has multiplicity $\leqq r+1$, then $\operatorname{ind} R \leqq r$.
E. Hewitt (Seattle, Wash.).

Smirnov, Yu. M. On coverings of topological spaces. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 204–215 (1951). (Russian)

Let R be a topological space. In this review, a covering of R means a finite open covering. A covering ω_2 follows a covering ω_1 if for every $G \in \omega_2$ there exists $H \in \omega_1$ such that $G \subset H$. The smallest cardinal number of a family of coverings of R which is cofinal, under this partial ordering, with the family of all coverings is called the combinatorial weight of R, and is denoted by the symbol $\sigma(R)$. The smallest cardinal number of an open basis for R is denoted by $\tau(R)$. The following results, all established by elementary arguments, are given first. 1) If R is a compact T_1-space, then $\sigma(R) = \tau(R)$. 2) The number of open-and-closed subsets of R is $\leqq \sigma(R)$. 3) If R is metrizable and non-compact, then $\sigma(R) \geqq 2^{\aleph_0}$. 4) A regular infinite space R is compact and metrizable if and only if $\sigma(R) = \aleph_0$. The Čech-Stone compactification βR for a completely regular space R is next taken up. This space is here written as αR, and its discovery is vaguely attributed to P. S. Aleksandrov, whose construction [Mat. Sbornik N.S. 5(47), 403–423 (1939); these Rev. 1, 318] is valid only for normal spaces. No mention is made of Wallman's previous work [Ann. of Math. (2) 39, 112–126 (1938)] giving an identical construction, or of the reviewer's extension of this construction to arbitrary completely regular spaces [Trans. Amer. Math. Soc. 64, 45–99 (1948); these Rev. 10, 126]. Let R^* be a space, R a dense subspace of R^*, and A a subset of R open in the relative topology of R. Let $O(A)$ be the union of all sets H^* open in R^* such that $R \cap H^* = H$. A number of obvious properties of this operator are set down. If R is normal and R^* is a compact Hausdorff space, then R^* is βR if and only if $\{O(U_j)\}_{j=1}^{n}$ is a covering of R^* whenever $\{U_j\}_{j=1}^{n}$ is a covering of R. For a normal R, the operator O maps the system of open-and-closed sets in R onto the system of open-and-closed sets in βR in a one-to-one fashion. For normal R, $\sigma(R) = \sigma(\beta R)$. If R and S are normal and S is a continuous image of R, then $\sigma(S) \leqq \sigma(R)$. From this, it follows that if R is normal and R^* is a compact Hausdorff space containing R as a dense subset, then $\sigma(R^*) \leqq \sigma(R)$. *E. Hewitt* (Seattle, Wash.).

Aleksandrov, P. On components of maximal bicompact extensions. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 216–218 (1951). (Russian)

Notation as in the preceding review. For normal R, the components of βR are in one-to-one correspondence with the family of all ultrafilters in the algebra of open-and-closed subsets of R. For a component Q of βR, Q is the intersection of the family $\{H_\gamma\}$ of all open and closed sets containing Q. The corresponding ultrafilter is $\{R \cap H_\gamma\}$. Other points are also discussed. *E. Hewitt* (Seattle, Wash.).

Smirnov, Yu. M. Mappings of systems of open sets. Mat. Sbornik N.S. 31(73), 152–166 (1952). (Russian)

Let Y and X be topological spaces. A mapping φ with domain the family of all open subsets of Y and with range contained in the family of open subsets of X is said to be a homomorphism of Y into X if (1) $\varphi(G \cap H) = \varphi(G) \cap \varphi(H)$ for all open subsets G and H of Y and (2) $\varphi(0) = 0$. Two sub-

The reviewer has misstated the definition of weak realization. In the definition given in the rev., "M is dense in M'" should be deleted.

sets A and B of a topological space X are said to be functionally separated if, for some real-valued continuous function f on X, $f(a)=0$ for $a \in A$ and $f(b)=1$ for $b \in B$. A finite family $\{G_i\}_{i=1}^m$ of open subsets of a topological space X is a normal covering of X if there exist sets A_i such that A_i and G_i' are functionally separated ($i=1, 2, \cdots, m$) and $\bigcup_{i=1}^m A_i = X$. A homomorphism φ of Y into X is normal if the image of every normal covering of Y is a covering of X. Every continuous mapping f of X onto Y produces an obvious homomorphism φ_f of Y into X: $\varphi_f(G)=f^{-1}(G)$, for all open subsets G of Y. The homomorphism φ_f is obviously normal. Given a homomorphism of Y into X, let $f_\varphi^{-1}(y)=\bigcap_{y \in G} \varphi(G)$. Given a continuous mapping f of X onto Y, it is clear that $f_{\varphi_f}=f$. The principal theorem proved is that if Y is a compact Hausdorff space and X is a completely regular space, then a homomorphism φ of Y into X is of the form φ_f for a continuous mapping f of X onto Y if and only if φ is a normal homomorphism. Various applications are discussed. *E. Hewitt* (Seattle, Wash.).

Katětov, M. On real-valued functions in topological spaces. Fund. Math. **38**, 85–91 (1951).

Consider the following properties of a topological space X. (A) For upper, resp. lower, semi-continuous real functions g and h on X such that $g \leqq h$, there exists a continuous f such that $g \leqq f \leqq h$. (B) For upper, resp. lower, semi-continuous real functions on X such that $g<h$ everywhere, there exists a continuous f such that $g<f<h$ everywhere. (C) Every countable open covering of X admits a locally finite refinement. (D) Every countable open covering of X admits a point-finite refinement. (E) For every countable open covering $\{G_n\}$ of X, there exist closed $F_n \subset G_n$ such that $\bigcup F_n = X$. (F) X is normal. The author proves that (F)→(A) and that (B), (C), (D), and (E) are equivalent. It is also proved that a bounded uniformly continuous function on a subspace of a uniform space admits a bounded uniformly continuous extension over the entire space. An initial lemma on which the proofs are based is incorrect; a corrected form of this lemma is to appear in Fund. Math. **40**. Hing Tong has announced [Bull. Amer. Math. Soc. **54**, 65 (1948)] and published a proof [see the following review] of the equivalence of (A) and (F). (The implication (A)→(F) is simple.) *E. Hewitt* (Seattle, Wash.).

Tong, Hing. Some characterizations of normal and perfectly normal spaces. Duke Math. J. **19**, 289–292 (1952).

The main results, announced already [Bull. Amer. Math. Soc. **54**, 65 (1948)], are as follows: a space is normal if and only if, given upper and, respectively, lower semicontinuous functions f, g with $f \leqq g$, there exists a continuous h with $f \leqq h \leqq g$; a space is perfectly normal if and only if every lower semicontinuous function is the limit of a monotonically increasing sequence of continuous functions. The "only if" part of the first of them is also contained in a note of the reviewer [see the preceding review]. [The reviewer wishes to point out that his note contains an incorrect lemma which may, however, be replaced by other lemmas without affecting the theorems and their proofs.] *M. Katětov* (Prague).

Cassina, Ugo. Su quattro proprietà equivalenti per i connessi irriducibili e sulla nozione di arco. Rend. Accad. Naz. dei XL (4) **2**, 139–153 (1951).

The author proves (without the axiom of choice and using elementary methods) that for subsets i of Euclidean n-space, irreducibly connected between two distinct points, the following four properties are equivalent. (1) i is closed. (2) i has geometric continuity (defined in terms of a natural order in i). (3) i is locally connected. (4) i has the extremal property (also defined in terms of the order). A fairly complete historical review and bibliography of the various characterizations of an arc is included. *M. E. Shanks.*

de Groot, J. Realisations under continuous mappings. Nederl. Akad. Wetensch., Proc. **53**, 1538–1547 = Indagationes Math. **12**, 483–492 (1950).

Let M and M' be topological spaces, and let f be a continuous map of M onto M'. The author defines a subset M^* of M to be a realization of M' in M under f if f maps M^* topologically onto M'. After pointing out the relative rarity of the existence of realizations and pointing out their connection with retracts, the author defines a subset $^*M$ of M to be a weak realization of M' in M under f if and only if $^*M$ is dense in M, $^*M'=f(^*M)$ is dense in M', and f is topological on $^*M$. In the main theorem of the paper, the author proves that any continuous mapping f of a compactum M is topological on a dense G_δ-subset S of M such that the set $f(S)$ is a dense G_δ in M'. Hence for a continuous map on a compactum a weak realization can always be found. After giving a collection of examples, the author closes with the conjecture that monotone interior maps f of plane Peano continua such that each $f^{-1}(x)$ is a Peano continuum always have realizations. *E. E. Floyd* (Charlottesville, Va.).

Floyd, E. E. On related periodic maps. Amer. J. Math. **74**, 547–554 (1952).

Let X be a finite-dimensional compact Hausdorff space with finitely generated integral Čech cohomology groups. Let T be a periodic mapping of X onto itself of period p^n, p a prime. Let L be the fixed-point set of T and Y the space of orbits of T. From earlier results of the author [Trans. Amer. Math. Soc. **72**, 138–147 (1952); these Rev. **13**, 673] and of Liao [Ann. of Math. (2) **56**, 68–83 (1952); these Rev. **14**, 73] the homology groups of X over R (the rationals) as well as the homology groups of L and Y over I_p (integers modulo p) are finitely generated. Consequently, Lefschetz number $\alpha(T, R)$ is defined and so are the Euler-Poincaré characteristics $\chi(X, R)$, $\chi(L, I_p)$, $\chi(Y, I_p)$. In the papers cited above, certain relations were shown to exist between the χ's. Using these results, the author now obtains the relation $\alpha(T, R)=\chi(L, I_p)$.

In the next part of the paper it is shown that there exists an integer $r(X)$ such that if T is a mapping of X of period $p>r(X)$, then for some i such that $p/i=q=$a prime, $\chi(X, R)=\chi(L_i, I_q)$, L_i being the fixed-point set of T^i. Assume now that X is a manifold and that T is such that $\dim L_i \leqq 1$ for each i. It is shown that, if $\chi(X, R)<0$, the period of T cannot exceed $r(X)$. (This is an extension of a theorem of the reviewer [Lectures in topology, Univ. of Michigan Press, pp. 159–190, 1941, especially pp. 159–164; these Rev. **3**, 133] in which it was assumed that T is analytic and that $\dim L_i \leqq 0$.)

In the final section the author considers a sequence of periodic mappings T_i of X of fixed period p^n, p a prime. Assuming that X is metric, locally compact, of dimension $n<\infty$ and lc^n, it is shown that if the T_i's converge in a suitable manner to a continuous mapping T, the fixed-point sets L_i converge to the fixed-point set L in a manner which guarantees, when X is compact, that, from a certain point on, the homology groups of the L_i's over I_p are isomorphic to those of L. Corollary for compact X: given a T of period p^n, there exists an $\epsilon>0$ such that if T_1 is of period p^n and if

T_1 ϵ-approximates T, then the homology groups of the fixed-point sets L and L_1 are isomorphic. (It seems to the reviewer that there should exist an ϵ independent of T. In any case, the theorem suitably modified would seem to provide, for metric spaces at least, a new proof of Newman's theorem which asserts that, roughly, a periodic mapping cannot have uniformly small orbits [see, for example, Smith, Ann. of Math. (2) 42, 446–458 (1941); these Rev. 2, 324].)

P. A. Smith (New York, N. Y.).

Anderson, R. D. On monotone interior mappings in the plane. Trans. Amer. Math. Soc. 73, 211–222 (1952).

A main purpose of the paper is to prove (I) that there exists a monotone interior map f of the plane onto itself such that each $f^{-1}(x)$ is a non-degenerate compact continuum. This theorem furnishes the solution of a problem outstanding for a number of years. The author states that a simple modification of the argument for I yields a monotone interior map f of the 2-cell onto itself such that each $f^{-1}(x)$ is non-degenerate. This furnishes a counterexample to a statement of Rožanskaya [Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 178–179 (1949); these Rev. 11, 196] to the effect that there does not exist an open at least one-dimensional mapping of the n-cell onto itself. The author promises to prove in a later paper that, in I, f can be constructed so that each $f^{-1}(x)$ is a pseudo-arc. The second main purpose of the paper is to prove that (II) there exists a monotone interior mapping of a one-dimensional continuous curve in the plane onto the plane. The proofs make heavy use of chaining methods for plane sets in a spirit roughly similar to recent work of Moise, Bing, and others. *E. E. Floyd.*

Whyburn, G. T. On k-fold irreducibility of mappings. Amer. J. Math. 74, 910–912 (1952).

A map $f: X \to Y$ is at least k-fold if $\text{card}\{f^{-1}(y)\} \geqq k$ for each $y \in Y$. Further, if $\text{card}\{X_0 \cap f^{-1}(y)\} \geqq k$ for each $y \in Y$, f is irreducibly k-fold on $X_0 \subset X$, but this fails for some $y \in Y$ if X_0 is replaced by a closed proper subset. Assume X and Y to be compact metric and $Y = f(X)$. Then, if f is at least k-fold, it is irreducibly k-fold if and only if the set of those $x \in X$ with $\text{card}\{f^{-1}f(x)\} = k$ is dense. If f is at least k-fold, it is irreducibly k-fold on some closed set provided it is finite-to-one. *A. D. Wallace* (New Orleans, La.).

Haupt, Otto. Bemerkung zu einem Abbildungssatz von Herrn Béla Sz.-Nagy. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 147–161 (1952).

Suppose S is the unit sphere $\{\|x\| = 1\}$ in E^n, X a compact subset of E^n, and FX the boundary of X. Suppose the origin 0 is interior to X and that for each $p \in S$, the ray from 0 through p contains a unique point $f(p)$ of FX. The author proves that the mapping f is Lipschitzian (his term: "dehnungsbeschränkt") if and only if the paratingent of FX fails to include 0. This extends a result of Vincze and Szüsz [Acta Sci. Math. Szeged 14, 96–100 (1951); these Rev. 13, 573], who dealt with convex X. Also included is another necessary and sufficient condition and an extension to Hilbert space. *V. L. Klee, Jr.* (Charlottesville, Va.).

Edrei, Albert. On mappings which do not increase small distances. Proc. London Math. Soc. (3) 2, 272–278 (1952).

If X is a metric space and if f is a map of X into X, then a point x of X is said to be a contraction [an expansion] {an isometry} point under f in case there exists a positive number $\mu(x)$ such that $y \in X$ with $\rho(x, y) \leqq \mu(x)$ implies $\rho(f(x), f(y)) \leqq \rho(x, y)[\geqq \rho(x, y)]$ $\{= \rho(x, y)\}$. The following four theorems are proved. (1) If X is a totally bounded metric space and if f is a map of X onto X which does not increase distances less than a fixed bound μ, then all distances less than μ are invariant under f. (2) If X is a denumerable compact metric space and if f is a map of X onto X such that every point of X is a contraction point under f, then every point of X is an isometry point under f and every point of X is periodic under f. (3) If X is a bounded convex subset of n-dimensional euclidean space and if f is a map of X onto X such that every point of X is a contraction point under f, then all distances are invariant under f. (4) If X is a bounded open subset of n-dimensional euclidean space and if f is a homeomorphism of X onto X such that every point of X is a contraction point under f, then all points of X are isometry points under f and the distance between any point of X and the boundary of X is invariant under f. It is conjectured that if X is a compact metric space and if f is a map of X onto X such that every point of X is a contraction point under f, then every point of X is an isometry point under f.

W. H. Gottschalk (Philadelphia, Pa.).

Nagumo, Mitio. A note on the theory of degree of mapping in Euclidean spaces. Osaka Math. J. 4, 1–9 (1952).

Let D be a bounded open set in E^m and $f_t(x)$ a continuous family of mappings of $\bar{D}$ into E. The author proves that if $a(t)$ is a point of E^m, $a(t)$ continuous, such that $a(t)$ non-ϵ $f_t(\bar{D} - D)$, then the degree of the map $f_t(D)$ at a is constant, $0 \leqq t \leqq 1$. The proof uses a special case of a theorem of Sard [Bull. Amer. Math. Soc. 48, 883–890 (1942); these Rev. 4, 153] and "an idea of Birkhoff and Kellogg" [Trans. Amer. Math. Soc. 23, 96–115 (1922)]. (The author attributes Sard's theorem to Sard and Kneser, but actually Kneser's results are not in terms of measure [Math. Z. 54, 34–51 (1951); these Rev. 12, 812].) The special case of Sard: Under a map of class C^r of D^{m+1} into E^m, the points $y = f(x)$ where rank $(\partial_x f) \leqq r$ have measure 0 if $r < m$.

If f is of class C^1 and $a = (a_1, \cdots, a_m)$ is not a critical value (image of a point where $\det(\partial_x f) = 0$), the degree is defined by the author as the number of times a is obtained with $\det(\partial_x f)$ positive less the number of times it is obtained with $\det(\partial_x f)$ negative. In this case the theorem is proved with the help of Sard's theorem, by considering the locus, shown to consist of curves, where $f_t(x) = a$. If put in the form $x = \chi(s)$, $t = \tau(s)$ with s = arc length in direction such that

$$\det \begin{pmatrix} \partial x_j f_t^i & \partial_t f_t^i \\ \overline{\chi}_j' & \tau' \end{pmatrix} > 0,$$

it is shown that the sign of $\tau'(s)$ = the sign of $\det(\partial_x f_t)$. Hence the degree for $t = 0$ $(t = 1)$ is the algebraic sum of the number of intersections of the curves above with the hyperplane $t = 0$ $(t = 1)$ each taken positive or negative according to the sign of τ'. With use of this result, the degrees of f_0 and f_1 are shown to be equal when f_t is of class C^2. When f_t is merely assumed to be continuous, the degree is defined in terms of approximating maps of class C^2. The proof uses the device of replacing the degree at $a(t)$ for the map $f_t(x)$ by the degree at the fixed point 0 for the map $F - a(t)$, where F is an approximation of class C^2 to f_t.

A. B. Brown (Flushing, N. Y.).

Ylitch-Dalovitch, Militsa. Une démonstration de la formule générale de Poincaré, concernant les surfaces fermées orientées dans un espace à trois dimensions. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 57–60 (1951). (Serbo-Croatian. French summary)

Watanabe, Sigekatu, and Oyama, Koichi. A combinatorial topological problem, concerning the theory of electrical networks. Rep. Univ. Electro-Commun. **1951**, no. 3, 61–69 (1951). (Japanese. English summary)

It is proved that, if a certain number of the edges of a closed connected network K are duplicated so that the number of edges starting from any vertex of K becomes even, K can be made a cycle, and the application of this result to the verification of the first law of Kirchhoff relative to such an electrical network is also discussed. (From authors' résumé.) *K. Iwasawa* (Cambridge, Mass.).

Rohlin, V. A. Intrinsic definition of Pontryagin's characteristic cycles. Doklady Akad. Nauk SSSR (N.S.) **84**, 449–452 (1952). (Russian)

The author gives a definition of Pontryagin's characteristic cycles of a differentiable manifold [Mat. Sbornik N.S. **21(63)**, 233–284 (1947); **24(66)**, 129–162 (1949); these Rev. **9**, 243; **10**, 727], which does not require an imbedding of the manifold and applies to a general (oriented) fiber bundle Π with a vector space R^n as fiber. Let E be the set of all systems of m vectors in n-space, let P_σ be the set of such systems subordinated to one of the functions σ considered by Pontryagin (cf. the cited reviews for definitions), and let Q_σ be the (open) complement of P_σ. The homotopy groups of Q_σ are shown to be trivial below $r(\sigma)-1$ ($r(\sigma)=\sigma(0)+\cdots+\sigma(n-1)$), and in dimension $r(\sigma)-1$ are infinite cyclic, resp. of order two, if $n+i_0+i_p\equiv 0 \pmod 2$ ($p=1,\cdots,s-1$), resp. otherwise; a generator is determined. The original bundle Π determines a new bundle with Q_σ as fiber; in the obstruction cocycle $Y(\sigma)$ of dimension $r(\sigma)$ one has the characteristic σ-class of Pontryagin. The usual definition of Stiefel-Whitney classes is a special case, for a certain σ; the Q_σ is essentially the Stiefel manifold. A regular map of a manifold M^k into R^{k+l} determines the tangent and normal bundle Π_T and Π_N. A duality theorem is stated: with each function σ on $(0,\cdots,k-1)$ is associated a certain function σ' on $(0,\cdots,l-1)$, and one has $Y_T(\sigma)=Y_N(\sigma')$. *H. Samelson* (Princeton, N. J.).

Hilton, P. J. The Hopf invariant and homotopy groups of spheres. Proc. Cambridge Philos. Soc. **48**, 547–554 (1952).

Let $H\colon \pi_r(S^n)\to\pi_r(S^{2n-1})$ denote the generalized Hopf homomorphism introduced by G. W. Whitehead for the case $r<3n-3$ [Ann. of Math. (2) **51**, 192–237 (1950); these Rev. **12**, 847] and extended by the present author [Proc. London Math. Soc. (3) **1**, 462–493 (1951); these Rev. **13**, 674] to all cases where $r<4n-3$ and to some other cases as well. Let $\eta\in\pi_r(S^p)$, $\theta\in\pi_p(S^n)$, and $\theta\circ\eta\in\pi_r(S^n)$ be their composition. Assume that $H(\theta\circ\eta)$ is defined. The author's main results state that $H(\theta\circ\eta)=H(\theta)\circ\eta$ if η is the suspension of some element of $\pi_{r-1}(S^{p-1})$, and

$$H(\theta\circ\eta)=\pm(F^{n-1}\theta)\circ(F^{p-1}\theta)\circ H(\eta)$$

if θ is the suspension of an element of $\pi_{p-1}(S^{n-1})$. Here $F^k\theta$ denotes the k-fold suspension of θ. These results may be regarded as generalizations of Theorems 5.19 and 5.24 of G. W. Whitehead [loc. cit.].

The author applies these results in two directions: Firstly, he extends G. W. Whitehead's list [loc. cit.] of non-zero elements of homotopy groups of spheres. In particular he identifies certain non-zero elements of $\pi_7(S^4)$ and $\pi_8(S^4)$; since these groups are both of order two according to recent work of J. P. Serre [C. R. Acad. Sci. Paris **234**, 1340–1342 (1952); these Rev. **13**, 675], it follows that these are the only such non-zero elements. Secondly, he studies under what conditions the "modified distributive law" for the composition operation, $(\beta_1+\beta_2)\circ\alpha=\beta_1\circ\alpha+\beta_2\circ\alpha+[\beta_1,\ \beta_2]\circ H(\alpha)$, holds true; here $\beta_1,\beta_2\in\pi_r(X)$ and $\alpha\in\pi_n(S^r)$. G. W. Whitehead proved this formula is always true for $n<3r-3$ [loc. cit.], and it has since been proven true for $n<3r-1$. The author proves that the formula remains valid for $3r-2<n<4r-3$ provided α can be "factored", $\alpha=\theta\circ\eta$, $\theta\in\pi_p(S^r)$, $\eta\in\pi_n(S^p)$, θ or η is the suspension of some element, and certain dimensional restrictions hold. Apparently no one has yet exhibited an exception to this modified distributive law.

The paper ends with some special results. It should be pointed out that Theorem 4.4 is a special case of Satz IV of a paper by B. Eckmann [Comment. Math. Helv. **14**, 234–256 (1942), these Rev. **3**, 318], and that Theorem 4.5 is also contained in this same paper by Eckmann. *W. S. Massey* (Providence, R. I.).

Adem, José. The iteration of the Steenrod squares in algebraic topology. Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952).

Throughout this review, $H^q(K)$ will denote the q-dimensional cohomology group of a complex K with integers modulo 2 as coefficients, and $\mathrm{Sq}^i\colon H^q(K)\to H^{q+i}(K)$, $i=0,1,2,\cdots$, will denote the Steenrod squares [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154]. The author's main result is the following set of relations between various iterated Steenrod squares: For all integers $s>t$, $\mathrm{Sq}^a\circ\mathrm{Sq}^b=\sum_{j=0}^{t}\binom{a-j-1}{a-2j}\mathrm{Sq}^{a+b-j}\circ\mathrm{Sq}^j$, where $a=s-t-1$, $b=s+t$, and $\binom{r}{q}$ is the binomial coefficient reduced modulo 2 with the convention that $\binom{r}{q}=0$ if $q>r$.

This relation has many interesting consequences. Some of those listed by the author are as follows: (1) Any Steenrod square can be written as a sum of iterated Steenrod squares whose exponents are all powers of two. (2) Some formulae are derived which impose strong restrictions on self-products in the cohomology ring of a complex. For example, if u is an integral 7-dimensional cohomology class on the complex K, and $u\cdot u\neq 0$ (cup product), then $H^{11}(K)\neq 0$. (3) The author studies the question of the existence of a connected, orientable, $2k$-dimensional manifold M whose k-dimensional (rational) Betti number is one. If we assume that $k=2^r(2r+1)$, $r>0$, (i.e., k is not a power of 2) and $H^q(M)=0$ for $q=2^r$, and $q=k-2^p$, $p=1,\cdots,n-1$, then such a manifold cannot exist. This result partially solves a problem considered by G. Hirsch [Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12, Paris, 1949, pp. 35–42; these Rev. **11**, 610] and A. Bassi [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **6**, 669–714 (1935)]. (4) The Hopf invariant of a map $S^{2n-1}\to S^n$ must be even if n is not a power of two. This partially solves a problem proposed by H. Hopf [Math. Ann. **104**, 637–665 (1931); Fund. Math. **25**, 427–440 (1935)]. Whether or not maps of Hopf invariant one always exist in case n is a power of two, remains an open, and apparently very difficult, question. As a corollary of this result, it follows that an m-sphere S^m admits a representation as a sphere bundle over S^n only if $m=2n-1$ and n is a power of two, and a continuous multi-

plication with a two-sided identity can be defined in S^m only if $m+1$ is a power of two. (5) Under certain conditions the composition of a map of Hopf invariant one with the k-fold suspension of a map of Hopf invariant one must be essential. In particular, the author identifies certain non-zero elements of $\pi_{n+8}(S^n)$ for $n>3$ and of $\pi_{n+14}(S^n)$ for $n>7$. (6) The relations obtained for iterated squares can be used to construct new cohomology operations of the "second kind", i.e., cohomology operations which are defined only on the kernel of a certain Steenrod square. These new cohomology operations can be used to give an effective procedure for deciding the homotopy class of a given simplicial map $S^{n+2}\to S^n$, and for determining the $(n+3)$-dimensional obstruction to the extension of a given map of the n-skeleton of a complex into an n-sphere (assuming, of course, that the map is extendable over the $(n+2)$-skeleton).

The author concludes this note with a brief indication of his methods. He promises that full details will appear elsewhere. *W. S. Massey* (Providence, R. I.).

Boltyanskiĭ, V. Cross-sections of fiber bundles. Doklady Akad. Nauk SSSR (N.S.) **85**, 17–20 (1952). (Russian)

Let P be a fiber bundle with base space a complex B, structure group a compact Lie group Γ, transitive over the fiber C, with connected isotropy group Γ_1. Suppose $\pi_1(C)=\cdots=\pi_{r-1}(C)=0$ [if $r=1$, then $\pi_1(C)$ is abelian and operates trivially on $\pi_2(C)$]. The author gives a formula for the possible second obstructions of cross-sections over B^{r+1}, assuming that the first obstruction vanishes:

$$Z^{r+2}=Z_{\mathfrak{S}}{}^{r+2}+R^{r+2}(D^r)+D^r\cup Y_{\mathfrak{S}}{}^2.$$

Here $Z_{\mathfrak{S}}{}^{r+2}$ is the second obstruction of an arbitrarily chosen cross-section $\mathfrak{S}$. D^r is the difference cocycle of an arbitrary map f (extendable to B^{r+1}) and the constant map of B^r into C, $R^{r+2}(D^r)$ is the second obstruction of f. $Y_{\mathfrak{S}}{}^2$ is a 2-cocycle with coefficients in $\pi_1(\Gamma_1)$, which for $r>2$ does not depend on $\mathfrak{S}$, and generalizes the second Whitney class of a sphere bundle; it is the primary obstruction of the principal bundle with group Γ_1, over B^{r+1}, determined by the cross-section $\mathfrak{S}$ over B^{r+1}. In the third term the coefficients $\pi_r(C)$ and $\pi_1(\Gamma_1)$ are paired to $\pi_{r+1}(C)$ (with μ: $S^1\to\Gamma_1$ and ν: $E^r\to C$, $\nu(\dot{E}^r)=q=$fix point of Γ_1, construct λ: $(E^r\times E^1)^{\cdot}\to C$ by $\lambda(b,a)=\mu(a)\cdot\nu(b)$ for $(b,a)\in E^r\times S^1$, $\lambda(b,a)=q$ for $(b,a)\in\dot{E}^r\times E^1$). The term $R^{r+2}(D^r)$ has been discussed by Postnikov [same Doklady (N.S.) **71**, 1027–1028 (1950); these Rev. **11**, 676]. The formula generalizes the author's formula for sphere bundles [ibid. **80**, 305–307 (1951); these Rev. **14**, 74]. Hopf's results [Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12, Paris, 1949, pp. 55–59; these Rev. **11**, 610] on fields of 2-contact elements in 4-manifolds follow.

H. Samelson (Princeton, N. J.).

Weil, André. Sur les théorèmes de de Rham. Comment. Math. Helv. **26**, 119–145 (1952).

Démonstration datant de 1947 des théorèmes de de Rham [J. Math. Pures Appl. (9) **10**, 115–200 (1931)]. C'est une des sources de la théorie de l'homologie de H. Cartan [Notes de cours miméographiées: Algebraic topology, Harvard Univ., 1948; ces Rev. **11**, 46; Séminaires de topologie algébrique de l'Ecole Norm. Sup., 1948–49, 1950–51]. La technique repose sur l'emploi du nerf N d'un recouvrement différentiablement simple (r.d.s.) d'une variété différentiable paracompacte V, de dimension n. Un r.d.s. est un recouvrement $\mathfrak{U}$ localement fini de V par des ouverts relativement compacts dont toute intersection non vide possède une rétraction différentiable. Il existe, pour chaque dimension, des isomorphismes canoniquement déterminés par $\mathfrak{U}$ entre: 1) le groupe de de Rham de V et le groupe de cohomologie de N; 2) le groupe d'homologie singulière différentiable de V et le groupe d'homologie des chaînes de N, à coefficients dans un groupe abélien. Enfin les relations de dualité entre les groupes de de Rham et les groupes d'homologie singulière à coefficients réels de V sont les mêmes qu'entre les groupes de cohomologie et d'homologie de N. D'où, en particulier, les théorèmes de de Rham sur les variétés compactes.

En utilisant la trivialité de l'homologie des ouverts de $\mathfrak{U}$ modulo leur frontière, sauf en dimension n (propriété établie pour les coefficients réels à l'aide des formes différentielles), on démontre le théorème de dualité de Poincaré à coefficients réels. Les ouverts de $\mathfrak{U}$ étant homotopiquement triviaux, N a même type d'homotopie que V. Plus généralement: Soit E un espace tel que $E\times E\times[0,1]$ soit normal et $\mathfrak{U}$ un recouvrement localement fini de E par des ouverts dont toute intersection non vide possède la propriété d'extension, alors, le nerf de $\mathfrak{U}$ a même type d'homotopie que E.

P. Dolbeault (Paris).

Ellis, David. On immediate inclusion in partially ordered sets and the construction of homology groups for metric lattices. Acta Sci. Math. Szeged **14**, 169–173 (1952).

Ce travail ne contient aucun élément nouveau, la relation "d'inclusion immédiate" ayant été définie pour une relation acyclique et étudiée par le rapporteur [Bull. Soc. Math. France **76**, 114–155 (1948); ces Rev. **10**, 502; Thèse, Paris, 1951], la possibilité d'une théorie de l'homologie pour des treillis avec valuation étant connue depuis longtemps [cf. les travaux de Tucker en particulier] et signalée dans la seconde édition du "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; ces Rev. **10**, 673] de G. Birkhoff que l'auteur cite pourtant. Le rapporteur se permet de signaler à cette occasion que la notion de complexe semi-simplicial, due à Eilenberg et Zilber [Ann. of Math. (2) **51**, 499–513 (1950); ces Rev. **11**, 734] semble fort utile pour une approche latticielle de l'homologie.

J. Riquet (Paris).

GEOMETRY

***Altshiller-Court, Nathan. College geometry. An introduction to the modern geometry of the triangle and the circle.** 2d ed. Barnes & Noble, Inc., 1952. xix+313 pp. $4.00.

Petričević, Feodor. Die Lösung des Apollonischen Problems durch stereographische Projektion. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 92–97 (1952). (Serbo-Croatian. German summary)

Walker, A. W. The differential equation of a conic and its relation to the aberrancy. Amer. Math. Monthly **59**, 531–538 (1952).

Ulčar, Jože. Geometrische Konstruktion einer quadratischen Korrespondenz zwischen Geraden von zwei Ebenen. Bull. Soc. Math. Phys. Macédoine **2**, 117–120 (1951). (Macedonian. German summary)

Rios de Souza, Jayme Eduardo. Pohlke's theorem and its generalization to the case of central projection. Anais Fac. Ci. Porto **35**, 197–216 (1951). (Portuguese)

Kok, J. C. A projective generalization of the nine-point circle and some properties connected with it. Nieuw Tijdschr. Wiskunde **40**, 1–14 (1952). (Dutch)

The author generalizes the nine-point circle of a triangle ABC by taking an arbitrary line l and point U to play the roles of the line at infinity and the orthocentre. Letting T denote the trilinear pole of l, and p the trilinear polar of U, he finds that the following sixteen points (of the complex projective plane) all lie on one conic: six points such as $AT\cdot BC$ and $AU\cdot BC$, six such as the harmonic conjugate of $AT\cdot p$ with respect to A and T and the harmonic conjugate of $AU\cdot l$ with respect to A and U, and finally the invariant points X, Y, and Z, W of the involutions determined on l and p by the respective quadrangles $ABCU$ and $ABCT$. He observes that this sixteen-point conic remains the same when the four points A, B, C, U are permuted while l is unchanged. He gives a projective proof for the following version of Feuerbach's Theorem: the sixteen-point conic touches the four conics through X and Y which touch the sides of the triangle ABC. *H. S. M. Coxeter.*

***Herrmann, Horst. Übungen zur projektiven Geometrie.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe, Band 18. Verlag Birkhäuser, Basel, 1952. i+168 pp. 17.00 Swiss francs.

This book is a kind of supplement to W. Blaschke's Projektive Geometrie [3rd ed., Birkhäuser, Basel-Stuttgart, in press; for a review of the 1st ed. see these Rev. **10**, 58], making use of techniques developed by H. F. Baker [Principles of geometry, vols. 1–6, Cambridge, 1922–1933] and H. G. Forder [The calculus of extension, Cambridge, 1941; these Rev. **3**, 12]. The author stresses the importance of matrices, pointing out that a matrix of three rows and three columns can represent the vertices or sides of a triangle, the coefficients in the equation of a conic (locus or envelope), a collineation, or a correlation, and that a matrix of four rows and four columns is analogously useful in solid geometry. In this manner he establishes the background for a systematic study of configurations along the lines of his vollständige regelmässige Konfigurationen [Arch. Math. **2**, 207–215 (1950); these Rev. **11**, 736]. The configurations described include projective generalizations of the Platonic solids, and "binomial" configurations which are projections of sections of simplexes in higher spaces. He calls two figures "dualistic" if the points and lines of one correspond to the lines and points of the other, and "dual" if the points and planes of one correspond to the planes and points of the other. The reader's understanding of such configurations is greatly assisted by the set of four stereoscopic figures supplied with a pair of red and green spectacles in a pocket at the back of the book.

There is a small error in Section 5. More seriously, in Section 158 the author describes the three desmic tetrahedra as being cyclically perspective, whereas the relevant permutations of the vertices of the first tetrahedron really form a 4-group and not a cyclic group [cf. Coxeter, Bull. Amer. Math. Soc. **56**, 413–455 (1950), p. 453; these Rev. **12**, 350]. *H. S. M. Coxeter* (Toronto, Ont.).

***Zacharias, Max. Das Parallelenproblem und seine Lösung. Eine Einführung in die hyperbolische nichteuklidische Geometrie.** 2d. ed. B. G. Teubner, Verlagsgesellschaft, Leipzig, 1951. 42 pp.

Zacharias, Max. Konstruktionen der ebenen Konfigurationen $(12_4, 16_3)$. Math. Nachr. **8**, 1–6 (1952).

The author points out that there are five essentially different configurations of twelve points lying by threes on sixteen lines in a plane. In two previous papers [Deutsche Math. **6**, 147–170 (1941); Math. Nachr. **2**, 163–170 (1949); these Rev. **8**, 219; **11**, 124], the author gave linear constructions (in the plane) for the configurations of Hesse [J. Reine Angew. Math. **36**, 143–176 (1848)] and de Vries [Acta Math. **12**, 63–81 (1889)]. To these he now adds constructions for the configurations of Bydžovský [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1939**, no. 2 (1940); these Rev. **7**, 390] and Metelka [ibid. **1944**, no. 21 (1946); these Rev. **7**, 390] and for his own new configuration [Math. Nachr. **1**, 332–336 (1948); these Rev. **10**, 563]. *H. S. M. Coxeter.*

Cundy, H. Martyn. 25-point geometry. Math. Gaz. **36**, 158–166 (1952).

It is observed that we may use $GF(5)$ for coordinates in a finite Euclidean plane and that this can be extended to a projective plane. Properties of triangles, conic sections, and other familiar geometrical figures are considered. The writer assumes that Pascal's theorem is a consequence of the axioms for projective planes. Though not in general true, this conclusion does hold in the finite plane considered here.

Marshall Hall (Columbus, Ohio).

Kelly, P. J., and Paige, L. J. Symmetric perpendicularity in Hilbert geometries. Pacific J. Math. **2**, 319–322 (1952).

Eine reelle einfach geschlossene konvexe kurve C, die höchstens ein gradliniges Stück besitzt, definiert eine Hilbertsche Metrik. Ist $h(P, Q)$ der entsprechend wie in der nicht-Euklidischen Geometrie definierte Hilbertsche Abstand der Punkte P, Q im Innern von C, so heisst F der Fusspunkt von P bezüglich der C treffenden Geraden g, wenn $h(P, F)\leqq h(P, X)$ für alle Punkte X auf g; die Gerade k heisst senkrecht zu g, wenn der Schnittpunkt von g und k Fusspunkt aller Punkte von k bezüglich g ist. Es wird bewiesen: die Beziehung des Senkrechtstehens ist genau dann eine in g und k symmetrische Beziehung, wenn C eine Ellipse ist. Zum Beweis wird ein Satz aus einer demnächst erscheinenden Arbeit von H. Busemann und P. Kelly übernommen, der ein hinreichendes Kriterium dafür, dass k auf g senkrecht steht, in polarentheoretischer Form gibt. Ferner benutzen Verff. eine elementargeometrische Charakterisierung der Ellipse von Tadahiko Kubota [Tôhoku Math. J. **9**, 148–151 (1916)], nach der eine differenzierbare konvexe Kurve eine Ellipse ist, wenn die Verbindung zweier Kurvenpunkte, in denen die Tangenten parallel sind, die Schnittpunkte der zu den Tangenten parallelen Sehnen mit der Kurve in konstantem Verhältnis teilt. *R. Moufang.*

Convex Domains, Extremal Problems, Integral Geometry

Drandell, Milton. Generalized convex sets in the plane. Duke Math. J. **19**, 537–547 (1952).

Sur le plan analytique (fermé par l'introduction d'un point à l'infini ω), on considère une famille $\{C\}$ de courbes fermées simples de Jordan, passant toutes par le point ω, et telle qu'il existe une courbe unique passant par deux points donnés distincts à distance finie. A partir de $\{C\}$, on définit une notion de convexité qui possède toutes les propriétés de la convexité ordinaire, définie à partir des droites (en particulier la frontière d'un ensemble fermé, borné, convexe, admettant des points intérieurs est une courbe de Jordan). Les démonstrations ne sont nullement immédiates, contrairement à ce qu'un examen superficiel de la question pourrait amener à penser. *J. Favard* (Grenoble).

Rado, R. A theorem on sequences of convex sets. Quart. J. Math., Oxford Ser. (2) **3**, 183–186 (1952).

By extending the author's proof with a familiar type of diagonal argument, the following result can be established: Every sequence S_n of sets admits a subsequence T_n which has exactly one of the following properties: (i) there is an integer k such that $\bigcap_1^k T_{a_i} \neq \Lambda = \bigcap_1^{k+1} T_{a_i}$ whenever $1 \leqq a_1 < \cdots < a_{k+1}$; (ii) $\bigcap_1^j T_{a_i} \neq \Lambda$ whenever $j \leqq a_1 < \cdots < a_j$. The proof is based on the following theorem of Ramsey [Proc. London Math. Soc. (2) **30**, 264–286 (1929)]: If all sets of r positive integers are divided into a finite number of classes, there is an infinite set of positive integers all of whose r-membered subsets are in the same class. When the sets S_i are convex subsets of E^n (the case actually treated by the author), it follows from Helly's theorem [Über. Deutsch. Math. Verein. **32**, 175–176 (1923)] that in case (i), $0 \leqq k \leqq n$, and in case (ii), $\bigcap_{n+1}^N T_i \neq \Lambda$ for each $N \geqq n+1$. There is also given a "finitist" version of this result, and a demonstration that for arbitrary $0 \leqq k \leqq n$, the case (i) can arise with convex S_i's in E^n. *V. L. Klee, Jr.*

Levi, F. W. Über zwei Sätze von Herrn Besicovitch. Arch. Math. **3**, 125–129 (1952).

Let K be a bounded convex body in Euclidean n-space and let $|K|$ be its volume. Let A, B, C be bodies with midpoints such that $A \subseteq K$, $B \supseteq K$, $C \supseteq K$ and C is convex; let $a(K) = \max |A|/|K|$, $b(K) = \min |B|/|K|$, and $c(K) = \min |C|/|K|$; let $a_n = \inf_K a(K)$, $b_n = \sup_K b(K)$, and $c_n = \sup_K c(K)$. Besicovitch [J. London Math. Soc. **23**, 237–240 (1948); these Rev. **10**, 320] proved $a_2 = 2/3$ and $b_2 = 4/3$. (The paper notes that this reviewer wrongly asserted that Besicovitch proved $c_2 = 4/3$.) This note shows: $a_n = 2 - b_n$; $c_2 = 2$, and $c(K) < 2$ unless K is a triangle; $a_n \geqq 2/(1+n^n)$; and $c_n \leqq n^n$. If $\mathfrak{A}, \mathfrak{B}$ are affine classes of bodies in the plane, define $\alpha(\mathfrak{A}, \mathfrak{B}) = \min |A|/|B|$, with $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, and $A \supseteq B$, and define a distance function by $|\mathfrak{A}, \mathfrak{B}| = \frac{1}{2} \log \alpha(\mathfrak{A}, \mathfrak{B}) \alpha(\mathfrak{B}, \mathfrak{A})$. It is proved that these affine classes make up a compact metric space Σ. Also every class $\mathfrak{A}$ ($\mathfrak{M}$) of bodies (with midpoints) lies within $\log 2$ ($\frac{1}{2} \log 2$) of the class of triangles (parallelograms). *M. M. Day.*

Favard, J. Sur quelques problèmes de couvercles. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 37–49. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

Let C be a convex body in ordinary space with surface area s^2 and volume v^3. Define $\sigma_s(C, d)$ to be the area of an orthogonal section of the supporting cylinder (cylinder of projection) of C in the direction d, and $\sigma_i(C, d)$ to be the maximal area of plane sections of C orthogonal to d. Maximize σ_s, σ_i over all d and over all C keeping s, v fixed. By circular symmetrization around an axis it is shown that σ_s, σ_i have a common maximum πR^2, where $R = R(s, v)$ and R/v is an increasing function of s/v, this maximum being attained only for a spherical lens. Also σ_s, σ_i have a common minimum πr^2, where $r = r(s, v)$ and r/v is a decreasing function of s/v, this minimum being attained only for a capsule (hemispherical capped cylinder, sphère étirée). Moreover, the perimeters λ_s, λ_i corresponding to σ_s, σ_i have $2\pi r$ as common minimum. In connection with this perimetric minimum the author proposes and purports to solve the following interesting problem: show that a sphere of radius r cannot be pushed through an inextensible flexible loop of perimeter λ unless $\lambda \geqq 2\pi r$. However, the reviewer questions the mathematical formulation of this problem and proof thereof. The paper concludes with a discussion of some analytical aspects of a general class of problems in the plane similar to the particular problems in space considered above. *W. Gustin.*

Dirac, G. A. Ovals with equichordal points. J. London Math. Soc. **27**, 429–437 (1952).

Let K be a closed convex curve. A point is called an equichordal point of K if every line passing through it meets K in two points a constant distance apart. The author investigates the problem whether there exists a K having two equichordal points. The author proves among other things that no closed convex curve exists with more than two equichordal points, and that if K has two equichordal points, then it has a unique tangent at every point. Incidentally the author attributes the problem of the nonexistence of a closed convex curve with two equichordal points to the reviewer; this is not correct. The reviewer is unable to trace to whom the problem is due. *P. Erdős* (Los Angeles, Calif.).

Sydler, J.-P. Sur les conditions nécessaires pour l'équivalence des polyèdres euclidiens. Elemente der Math. **7**, 49–53 (1952).

It is proved that if two polyhedra P and P' satisfy the necessary conditions for equivalence by decomposition due to Dehn [Math. Ann. **55**, 465–478 (1902)], then there exist polyhedra Q and Q' whose dihedral angles are rational multiples of π such that $P+Q$ and $P'+Q'$ are equivalent by decomposition. The reviewer was unable to follow all details of the proof, which is based on previous results of the author [Comment. Math. Helv. **16**, 266–273 (1944); these Rev. **6**, 183] and Hadwiger [Comment. Math. Helv. **24**, 204–218 (1950); these Rev. **12**, 526]. *B. Jessen* (Copenhagen).

Hadwiger, Hugo. Mittelpunktspolyeder und translative Zerlegungsgleichheit. Math. Nachr. **8**, 53–58 (1952).

Two polyhedra A and B of three-dimensional Euclidean space are called "translativ zerlegungsgleich" if $A = \sum_1^n A_\nu$, and $B = \sum_1^n B_\nu$, where A_ν and B_ν are polyhedra such that A_ν may be carried into B_ν by a translation. It is proved that a convex polyhedron is translativ zerlegungsgleich to a cube if and only if it is point-symmetric and has point-symmetric faces. *B. Jessen* (Copenhagen).

Hadwiger, H. Ergänzungsgleichheit k-dimensionaler Polyeder. Math. Z. **55**, 292–298 (1952).

Two polyhedra A and B of k-dimensional Euclidean space R_k are called "G-zerlegungsgleich", where G is a group of movements in R_k (notation: $A \sim B$) if $A = \sum_1^n A_\nu$ and $B = \sum_1^n B_\nu$, where A_ν and B_ν are polyhedra such that A_ν may

be carried into B, by a movement of G. They are called "G-ergänzungsgleich" if $A+C\sim B+D$, where $C\sim D$, and "G-selbstergänzungsgleich" if $n\cdot A\sim n\cdot B$ for some integer $n>1$. As a generalization of a result of Sydler [Comment. Math. Helv. 16, 266–273 (1944); these Rev. 6, 183] it is proved that if A and B are G-ergänzungsgleich, then they are also G-zerlegungsgleich. Moreover it is proved that if A and B are G-selbstergänzungsgleich, then they are also G-zerlegungsgleich. *B. Jessen* (Copenhagen).

Hadwiger, H. Einlagerung kongruenter Kugeln in eine Kugel. Elemente der Math. 7, 97–103 (1952).

For a positive number R, let $N(R)$ denote the greatest possible number of solid spheres of unit radius that can be packed inside a hollow sphere of radius R; e.g., $N(R)=0$ if $R<1$, and $N(R)=1$ if $1\leqq R<2$. The author proves that the packing-density $N(R)/R^3$ is always greater than $1/8$, this being the value approached (though never attained) when R increases towards the value 2. For this purpose, he gives a new proof of the inequality $N(R)\geqq(\pi/3\sqrt{2})(R-1)^3$ [L. Fejes-Tóth, Lagerungsprobleme in der Ebene, auf der Kugelfläche und im Raum, Springer, Berlin, in press], which shows that the packing-density is greater than $1/8$ for every $R>2.235\cdots$. Since any sphere large enough to contain five unit spheres will actually accommodate six, he soon finds that $N(2.235\cdots)=4$, and the proof is completed by investigating the smallest values of R for which $N(R)=3$ or 4. *H. S. M. Coxeter* (Toronto, Ont.).

Boerdijk, A. H. Some remarks concerning close-packing of equal spheres. Philips Research Rep. 7, 303–313 (1952).

Some configurations of equal spheres are described which are, according to certain criteria in local regions, closer than the configuration known as close-packing. One of these configurations disproves a conjecture of the reviewer according to which if a sphere of unit diameter is touched by 12 others, the center of a 14th sphere will have a distance from the center of the central sphere >1.38. In the configuration mentioned above the distance in question is $7/27^{1/3}=1.347$. Finally, a proof is given for the fact that the maximal number of spheres simultaneously touching a sphere is twelve. Since the proof rests on the supposition that each peripheral sphere is surrounded by exactly 5 others, the proof seems to the reviewer to be incomplete. On the other hand two exact proofs were given some time ago by Schütte and van der Waerden [Math. Ann. 123, 96–124 (1951); these Rev. 13, 61]. *L. Fejes Tóth* (Veszprém).

Whyte, L. L. Unique arrangements of points on a sphere. Amer. Math. Monthly 59, 606–611 (1952).

This is a survey of works dealing with spherical point systems determined by a certain extremum property. In particular, the following two problems are considered: to determine the stable equilibrium of N electrons free to move on a sphere, and the arrangement of N points on a sphere which maximize the least distance between any pair. *L. Fejes Tóth* (Veszprém).

Algebraic Geometry

Stubban, John Olav. Sur les systèmes linéaires réductibles de courbes algébriques planes. Mat. Tidsskr. B. 1951, 52–55 (1951).

Le présent mémoire apporte un raffinement à une inégalité de K. Lage Sundet [Mat. Tidsskr. B. 1949, 18–24; ces Rev. 10, 734]: Un système linéaire de courbes algébriques de genre virtuel $p\geqq 0$ et de dimension virtuelle $r\geqq 0$, réductible, tel que $r>p-1$, contient une courbe fixe cremoniennement équivalente à une droite. Ce résultat tient au caractère additif de l'invariant $r-p+1$ dans l'addition des systèmes linéaires, et au fait que dans un système composé au moyen d'un faisceau de courbes de genre w, la condition $p\geqq 0$ entraine $w\geqq 1$ et $r-p+1\leqq 0$. Le mémoire se termine en montrant par un contre-exemple que l'inégalité obtenue ne peut plus être améliorée: si à un faisceau de Halphen H on ajoute la cubique Γ définie par ses 9 points bases, on a un faisceau $|H+\Gamma|$ qui admet comme composante fixe Γ non rationnelle, et qui vérifie $r-p+1=0$. Ce résultat donne en outre un exemple de courbe décomposée, virtuellement elliptique, sans composante rationnelle. *L. Gauthier.*

Ladopoulos, P. D. Sur la métrique des courbes algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 469–477 (1952).

If C_n', C_n'' are two plane curves of order n, the angle between them is defined to be $-i/2$ times the logarithm of the cross ratio of C_n', C_n'', I, J, where I and J are the curves of the pencil $C_n'+\lambda C_n''$ which contain the circular points. The main theorem of the paper states that this angle is equal (mod π) to the sum of the n angles obtained by pairing off the asymptotes of C_n' with those of C_n''. Various theorems concerning asymptotes and circular curves follow easily from this. *R. J. Walker* (Ithaca, N. Y.).

Gaeta, Federico. Caratterizzazione delle curve origini di una catena di resti minimali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 387–389 (1952).

This is a continuation of a previous note [same Rend. (8) 12, 270–273 (1952); these Rev. 14, 79]. The main result is the following: families of skew curves of finite residual whose generic curve C is the origin of a chain of ρ arithmetically normal curves, each of which is the residual of its predecessor with respect to two surfaces of minimum order passing through it, are characterized by the condition $\mu_{\rho+1,1}>0$, where $\mu_{\rho+1,1}$ is the invariant previously introduced by the author [Ann. Mat. Pura Appl. (4) 31, 1–64 (1950); these Rev. 13, 156]. Incidently, the author gives examples of families of curves of finite residual whose generic member is reducible. *J. A. Todd* (Cambridge, England).

Brown, L. M. A configuration in five dimensions. J. London Math. Soc. 27, 471–475 (1952).

H. F. Baker [Principles of geometry, v. 3, Cambridge, 1923, pp. 169–170] showed that fifteen of the 27 lines on Clebsch's "diagonal" cubic surface

$$x_1+\cdots+x_6=x_1^3+\cdots+x_6^3=0$$

are its complete intersection with $x_1^5+\cdots+x_6^5=0$. The author obtains the analogous result that the V_3^{15}

$$x_1+\cdots+x_7=x_1^3+\cdots+x_7^3=x_1^5+\cdots+x_7^5=0$$

(in projective 5-space) contains 105 planes, which are its complete intersection with

$$x_1^7+\cdots+x_7^7=0.$$

Corresponding to the remaining twelve lines on the cubic surface, he finds, on V_3^{15}, 360 lines, which fall into 120 sets of three; each set of three all meet seven of the 105 planes. *H. S. M. Coxeter* (Toronto, Ont.).

d'Orgeval, B. A propos de la surface intersection de trois quadriques de S^3 contenant une octique de genre trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 462–468 (1952).

L'A. considère ici la surface normale d'ordre 8 et de genre 1 de l'espace à 5 dimensions contenant une courbe d'ordre 8 et de genre 3; elle contient par conséquent un système linéaire ∞^3 de telles courbes. Cette surface a déjà été considérée par Godeaux [Bull. Soc. Roy. Sci. Liège **14**, 332–341 (1945); ces Rev. **8**, 528], qui a trouvé sur la surface même une base composée d'une section hyperplane et d'une des ∞^3 courbes d'ordre 8; le déterminant de cette base ayant la valeur -32. L'A. démontre ici que la base considérée par Godeaux est une base minima, et que la surface en question ne possède aucune transformation birationnelle en elle-même. *E. G. Togliatti* (Gênes).

Vaccaro, Michelangelo. Dimensione, ordine e invarianti delle traiettorie di un particolare gruppo di omografie. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 453–477 (1950).

This paper is, in substance, a geometrical study of equivalence of nth degree equations $a_1x^{n-1}=0$ under transformations of the group $x'=ax+b$ $(a\neq 0)$; in other words, it investigates the properties of the 2-parameter group of S_n-collineations which leave invariant a given rational normal curve C^n and a given point O of C^n. A trajectory of this collineation group is the locus—generally a surface—of points equivalent to a given point P vis-à-vis the group. The author investigates (i) the order $n(n-1)$ of the trajectory surface of a general point P of S_n, (ii) the various special positions of P for which this trajectory reduces to a curve or a point, or is a surface or order less than $n(n-1)$, (iii) a fundamental set of explicit equations for the trajectory of P, and (iv) special trajectories which are ruled surfaces.

J. G. Semple (London).

Nollet, Louis. Sopra la serie di Severi d'una superficie algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 273–276 (1952).

It is shewn that on a surface without exceptional curves of the first kind the Zeuthen-Segre invariant I cannot have the value -3, i.e., that the series of equivalence of Severi, which is virtually that of the Jacobian sets of the everywhere finite simple integrals (and in this respect analogous to the canonical series on a curve), and whose order is $I+4$, cannot have order equal to unity. *P. Du Val.*

Roth, Leonard. Sulle V_3 algebriche che contengono un sistema lineare di superficie di genere lineare $p^{(1)}\leq 1$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 401–405 (1952).

The author classifies algebraic threefolds which contain a linear system of surfaces of linear genus $p^{(1)}\leqq 1$.

J. A. Todd (Cambridge, England).

Galafassi, Vittorio Emanuele. Sulle falde delle rigate astratte reali. Rivista Mat. Univ. Parma **3**, 65–73 (1952).

Examples are constructed to shew that a real algebraic surface, birationally equivalent to a ruled surface of any genus p, may have an arbitrarily large number of distinct real sheets. The method is to construct a family of real conics of the required genus, of which an arbitrarily large number consist of two conjugate imaginary lines; the continuously variable real conic of the system consists alternately of one and of no real circuit, and each sequence of configurations of the former kind (from one degenerate conic to the next) generates a complete sheet of the surface. As, however, the increase in the number of degenerate conics in the pencil increases the Zeuthen-Segre invariant I, it is found that a surface referable to a ruled surface of genus p, and having also a given value for I, has a maximum number of sheets, namely

$$\begin{aligned} p+1 &\quad \text{if} \quad I+4p\leqq 3 \\ [\tfrac{1}{2}I]+3p &\quad \text{if} \quad I+4p\geqq 2. \end{aligned}$$

P. Du Val (Bristol).

Franchetta, Alfredo. Sui modelli pluricanonici delle superficie algebriche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 293–308 (1950).

The following theorems are proved for surfaces of arithmetic genus $p_a>0$: (i) If the linear genus $p^{(1)}>2$, the tricanonical system is without base points. (ii) If $p^{(1)}>1$, the quadricanonical and all higher multiples of the canonical system are without base points. (iii) If $p^{(1)}>1$, the tricanonical system is compounded, if at all, with an involution of order 2, whose model surface is referable to a ruled surface. (iv) If $p^{(1)}>1$, the quadricanonical and all higher multiples of the canonical system are simple.

Thus every surface with $p_a>0$, $p^{(1)}>1$, has a simple quadricanonical model whose order and ambient space are what one expects from the ordinary formulae.

P. Du Val (Bristol).

Gallarati, Dionigi. Intorno a certe superficie algebriche aventi un elevato numero di punti singolari isolati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 344–347 (1951).

In this paper the author constructs equations of systems of algebraic surfaces of order n with μ_n distinct nodes,

(1) $\mu_n=\frac{1}{4}(n+1)(n-1)^2$ for n odd,

(2) $\mu_n=\frac{1}{4}(n+1)(n^2-4)+1$ for n even.

In each case the value of μ_n is a lower limit to the number of nodes that a surface of order n can have. The author also obtains the equation of a system of algebraic surfaces of odd order n with a triple point and in addition μ_n distinct nodes, where

(3) $\mu_n=\frac{1}{4}(n+1)(n^2-9)$.

It should be noted that for surfaces of order $n>5$, formula (3) yields a larger value of μ_n than formula (1), so that surfaces of odd order $n\geqq 5$ may have at least the number of distinct nodes defined by (3) and an additional triple point. For example, for $n=7$, $\mu_7=72$ by (1), 80 by (3).

The above is of especial interest because of a recent paper by Togliatti [Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 47–59 (1950); these Rev. **12**, 855]. Togliatti made an examination of all previous efforts to obtain a formula for the maximum number of distinct nodes of an algebraic surface of given order and concluded that all formulas previously derived are erroneous. He closed his paper with the observation that definite results cannot be obtained until further research has shown the existence of surfaces of order $n\geqq 5$ "with many nodes". The present paper does that and is, therefore, a distinct contribution to the solution of the maximum number problem. *T. R. Hollcroft.*

Gallarati, Dionigi Sulle superficie del quinto ordine dotate di punti tripli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 70–75 (1952).

Dans cette note, l'auteur détermine le nombre maximum de points doubles isolés que peut présenter une surface

algébrique F^5 de l'espace projectif S_3, qui possède déjà un ou plusieurs points triples, mais qui est dépourvue de courbe multiple. Une telle surface est un plan double, et les relations entre ses singularités et celles de la courbe de branchement sont déduites des résultats de G. Vaccaro sur les V^n ayant un point $(n-2)$-uple [mêmes Rend. (8) **3**, 288–293, 314–321 (1947); ces Rev. **9**, 610]. Si F^5 présente t points triples, le nombre maximum de points doubles correspondant $D(t)$ a pour valeurs: $D(1)=24$; $D(2)=20$; $D(3)=16$; $D(4)=4$; $D(5)=0$; et une surface F^5 a au maximum 5 points triples isolés.

Le résultat $D(1)=24$ est dû à E. Togliatti [Studi in onore del Professore Salvatore Ortu Carboni, Rome, 1935, pp. 253–259]. Pour $t\geqq 2$, l'existence des surfaces ayant le nombre maximum de singularités est établie par l'auteur, dans chaque cas, en étudiant les propriétés du cône des tangentes en un point triple, au moyen de la représentation elliptique de Weierstrass. Dès que $t\geqq 3$, on peut abaisser l'ordre de F^5 par une transformation birationnelle $x'=1/x$.

L. Gauthier (Nancy).

Tosi, Armida. Formule di Plücker e principio di corrispondenza. Period. Mat. (4) **30**, 134–152 (1952).

Vest, M. L. Cremona transformations associated with the chords of a twisted cubic. Duke Math. J. **19**, 397–408 (1952).

L'auteur poursuit dans ce mémoire l'étude des transformations de Cremona de S_3 dont les points homologues sont joints par les droites d'une congruence d'ordre un: Les cas où la courbe focale de cette congruence est décomposée ont été traités par l'auteur dans trois mémoires antérieurs [Bull. Amer. Math. Soc. **48**, 874–882 (1942), où sont examinées les transformations non involutives; Duke Math. J. **12**, 621–628 (1945); **13**, 401–409 (1946), où sont examinées les transformations involutives; ces Rev. **4**, 170; **7**, 326; **8**, 341]. Dans le présent mémoire, l'auteur traite le cas où la congruence est à focale irréductible (congruence des cordes d'une cubique gauche r). L'involution I est alors définie de la manière suivante: Soit $|F^{2n+2}|$ un faisceau linéaire de surfaces admettant r comme courbe multiple d'ordre n. Les surfaces du faisceau découpent sur les cordes de r les couples de l'involution. La transformation non involutive T est définie ainsi: Soient $|F^{2n+1}|$ et $|F'^{2m+1}|$ deux faisceaux linéaires de surfaces admettant r avec les multiplicités respectives n et m. Si l'on associe projectivement les deux faisceaux, deux points P, P' sont correspondants dans T lorsque la droite PP' est une corde de r et que les surfaces F^{2n+1} passant par P et F'^{2m+1} passant par P' sont homologues.

L'auteur étudie analytiquement, au moyen de la représentation rationnelle de r, les correspondances I et T en déterminant explicitement les équations de ces correspondances, les éléments fondamentaux, les surfaces unies, les réseaux homaloïdaux, les jacobiennes, et en déduit les tables d'intersection.

Remarque du reviewer: Les involutions I ainsi définies dans S_3 sont des généralisations de l'involution de Reye. Une autre généralisation dans S_3 est traitée par P. Mathieu dans sa thèse, dont les résultats sont condensés dans C. R. Acad. Sci. Paris **230**, 1132–1134, 1567–1569 (1950) [ces Rev. **11**, 614, 682]. On peut aisément démontrer que les involutions de M. L. Vest sont rationnelles: Il aurait été intéressant d'étudier leur représentation sur un S_3 et les surfaces de diramation de la correspondance (1, 2) ainsi mise en évidence, surfaces qui, dans le cas particulier $n=0$ de Reye, sont des surfaces de Kummer.

L. Gauthier.

Watanabe, Sigekatu. On birational transformations. Rep. Univ. Electro-Commun. **1951**, no. 3, 41–59 (1951). (Japanese. English summary)

The author considers three important properties of Cremona transformations, found by Cremona himself, and studies the classes of rational transformations of a plane into itself satisfying one, two, or three of these properties. It is shown that some properties of Cremona transformations are also valid for these general classes of rational transformations and, in particular, that Cremona transformations of order 2 are characterized by those three properties.

K. Iwasawa (Cambridge, Mass.).

Dedò, Modesto. Sulle condizioni di regolarità di una trasformazione analitica di De Jonquières. Boll. Un. Mat. Ital. (3) **7**, 108–110 (1952).

In the plane, three corresponding pairs of linear branches (γ_i, γ_i') $(i=1, 2, 3)$, with origins on Oy and tangents all distinct from Oy, suffice in general to define a unique generalized de Jonquières transformation, with y_∞ as centre, in a neighborhood of Oy. In this note the author shows how two of the given pairs (γ_i, γ_i') can be replaced by a single pair of second order branches—in particular, simple cuspidal branches—and discusses the conditions of regularity of the resulting transformation near $x=0$.

J. G. Semple.

Godeaux, Lucien. Remarques sur la représentation des transformations birationnelles planes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 836–849 (1951).

Ce mémoire expose une méthode de recherche de nouveaux réseaux homaloïdaux de courbes algébriques planes, qui résulte de la conjonction de deux travaux récents sur les transformations birationnelles planes: l'un du même auteur [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 2 (1949); ces Rev. **11**, 739] où la transformation T est représentée par une surface rationnelle normale F, l'autre de B. Segre [Ann. Mat. Pura Appl. (4) **28**, 107–139 (1949); ces Rev. **12**, 200] qui utilise systématiquement l'adjoint d'ordre $n-2$ à un réseau homaloïdal d'ordre n.

Soit F l'image de la transformation birationnelle plane T. L'adjoint $|D'|$ au système $|D|$ des sections hyperplanes de F est composé au moyen d'un faisceau si T est du type de de Jonquières. Sinon $|D'|$ est irréductible et son image F' est une surface rationnelle, qu'on représente sur un plan projectif ϖ au moyen d'un système linéaire $|\gamma|$. L'étude des ν points fondamentaux, supposés ordinaires, de T permet à l'auteur de montrer que F' est d'ordre $2n-\nu-1$, à sections de genre $n-\nu+2$ dans un S_{n-2}, et qu'aux deux réseaux homaloïdaux $|A_1|$ et $|A_2|$ de T, correspondent dans ϖ deux réseaux homaloïdaux $|\gamma_1|$ et $|\gamma_2|$ tels que $\gamma_1+\gamma_2\equiv 2\gamma-\gamma'$. Ce résultat permet de déterminer les transformations birationnelles T ayant un nombre donné de points fondamentaux: l'auteur expose cette détermination dans le cas où F est à sections elliptiques ($\nu=n+1$ et $6\leqq\nu\leqq 11$) et retrouve les transformations birationnelles obtenues par B. Segre. Puis il examine trois exemples où F' est à sections de genre 2, qui mettent en évidence six réseaux homaloïdaux intéressants ($\nu=n=8$, 9, et 13).

Le mémoire se termine en montrant la signification du processus inverse: Etant donnés deux réseaux homaloïdaux $|\gamma_1|$ et $|\gamma_2|$, qui, associés aux droites du plan, définissent deux transformations birationnelles T_1 et T_2, la surface F représentée sur le plan par le système complet $|\gamma_1+\gamma_2|$ est l'image d'une transformation birationnelle T, et on a $T=T_1\cdot T_2$.

L. Gauthier (Nancy).

Cossu, Aldo. Sulle trasformazioni puntuali tra spazi a rette caratteristiche coincidenti. Boll. Un. Mat. Ital. (3) **7**, 118–122 (1952).

L'auteur étudie les transformations ponctuelles T entre deux espaces ordinaires S_3, $\bar{S}_3$ au voisinage de deux points réguliers O, $\bar{O}$ correspondants, dans le cas de coïncidence des sept droites caractéristiques [Creangă, Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 177–227 (1940); ces Rev. **9**, 64; M. Villa, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 55–61, 192–196, 295–303 (1948); ces Rev. **10**, 210]. Ces sept droites sont, en général, les sept droites bases d'un réseau de cônes cubiques de sommet O. La note actuelle envisage le cas où les droites coïncident, soit celui où les cônes du système de cônes cubiques ci-dessus ont un contact du sixième ordre le long d'une génératrice base. Utilisant des résultats dûs à E. Bompiani [ibid. (8) **6**, 145–151 (1949), p. 150; ces Rev. **10**, 738] l'auteur établit les équations de la correspondance ponctuelle relative au cas envisagé, et la considération de la variété de Segre représentative des couples de points des deux espaces S_3, $\bar{S}_3$, lui permet de donner les équations des ∞^9 transformations cubiques osculatrices. *P. Vincensini.*

Burau, Werner. Grundmannigfaltigkeiten, ihre Dualitätstheorie und Fundamentalkorrelationen. Arch. Math. **3**, 130–136 (1952).

By a Grundmannigfaltigkeit (Gm.) is understood an algebraic manifold F_d in n-dimensional space P_n, having a continuous group $\mathfrak{G}$ of collineations into itself. It may also be transformed by a correlation into a family $\hat{F}_d$ of hyperplanes determined by F_d (e.g., its tangent hyperplanes). Such a correlation is called fundamental if it commutes with all the collineations of $\mathfrak{G}$; and Satz 1 states that if there is a fundamental correlation, it is unique, and involutory, i.e., its square is the identical collineation. The author concentrates on the case in which the points of F_d map an aggregate $\mathfrak{M}$ of geometrical elements E of any kind in k-dimensional space U_k, and the hyperplanes of $\hat{F}_d$ the dual aggregate $\hat{\mathfrak{M}}$ of elements $\hat{E}$ dual to E, $\mathfrak{G}$ being then isomorphic with the group of collineations in U_k which transform $\mathfrak{M}$, $\hat{\mathfrak{M}}$ into themselves. Satz 2 states that there is a fundamental correlation if and only if $\mathfrak{M}=\hat{\mathfrak{M}}$, each element E being self-dual. Of the special cases and corollaries considered the most interesting are:

1) If E consists of a hyperplane and a point lying in it, in U_k, F_d is a general hyperplane section of the Segre manifold, "product" of two P_h's; this manifold has a fundamental correlation. 2) A Segre manifold has a fundamental correlation if and only if all the spaces of which it is the "product" are lines. 3) The Grassmannian $G_{2,h}$ of h-dimensional subspaces in U_k has a fundamental correlation if and only if $k=2h+1$. *P. Du Val* (Bristol).

Burau, Werner. Geometrische Bemerkungen zu einigen Grundfragen der algebraischen Geometrie in idealtheoretischer Begründung. Monatsh. Math. **56**, 16–37 (1952).

The n-dimensional Veronese manifold V_n^h is defined parametrically in $\binom{n+h}{n}-1$ space by the equations

$$x_{i_0 i_1 \cdots i_n} = u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} \quad (i_0+i_1+\cdots+i_n=h).$$

Let $x=(\cdots, x_{i_0 i_1 \cdots i_n}, \cdots)$. The indices varying over all possibilities, the projective space spanned by the points x, $\partial x/\partial u_i$, $\partial^2 x/\partial u_i \partial u_j$, $\cdots$, $\partial^s x/\partial u_{i_1}\cdots\partial u_{i_s}$ (evaluated at $u_i=\bar{u}_i$) defines the sth osculating space $T^{(s)}$ of V_n^h at $x(\bar{u}_i)$. In §1, these osculating spaces are studied; in particular, for example, it is shown that two osculating spaces $T^{(p)}$, $T^{(q)}$ belonging to different points are skew if and only if $p+q<h$. These spaces may also be used to give a criterion for a hypersurface $f=0$ to have an s-fold point at a point P. To the hypersurface $f=0$ of degree k, where

$$f=\sum a_{i_0 i_1 \cdots i_n} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n},$$

is associated in P_{r-1}, $r=\binom{n+k}{k}-1$, the hyperplane $L(f)$: $\sum a_{i_0 i_1 \cdots i_n} x_{i_0 i_1 \cdots i_n}=0$; to a hypersurface $f=0$ of degree $t\leqq k$ is associated $L(f)=\bigcap_g L(f\cdot g)$, where g varies over all forms of degree $k-t$. It is shown that the point P of $f=0$ is at least $(s+1)$-fold if and only if $L^h(f)$ contains the osculating $T^{(s)}$ at the point corresponding to P. Similar criterion are given for P to be exactly s-fold. Several other basic properties of V_n^h are assembled in §1. In §2, these are applied to the representation of a manifold as the intersection of hypersurfaces. Let V^N be the mapping $u\to x(u)$; $V^N(M_d)$, the image of the d-dimensional manifold M_d; $\langle V^N(M_d)\rangle$, the projective space spanned by $V^N(M_d)$. Let P be a point of M_d. For all N sufficiently large, $\langle V^N(M_d)\rangle$ contains all the osculating spaces at $V^N(P)$ up to the sth, but only cuts out a proper subspace P_r from $T^{(s+1)}$. Necessary and sufficient for P to be simple is that $s=0$ and $r=d$. More generally the numbers s, r may be used to define the concept of "pure intersection". In the case of a manifold M_d represented as the intersection of hypersurfaces $f_1, \cdots, f_m$ of like degree, the condition of "pure intersection" amounts to saying that the Jacobian of $f_1, \cdots, f_m$ has rank $n-d$ at each point of M_d. The known theorem that a singularity-free manifold M_d in P_n can be represented as the pure intersection of at most $n+1$ hypersurfaces of like degree is deduced from the previous considerations on manifolds of Veronese. In §3, the results of §1 are used to define intersection multiplicity, and the definition is shown to be equivalent with that given in terms of lengths of primary ideals.

The author says that he will pursue algebraic geometry over the complex number field because that is the way it is usually done ["wie es meist üblich ist"]. Actually all his results go equally well for an arbitrary algebraically closed field of characteristic 0. At several places, the results, stated for an arbitrary point P, are verified for the point $(u)=(0, 0, \cdots, 1)$ only, but it is not explained why this is sufficient. *A. Seidenberg* (Berkeley, Calif.).

Severi, Francesco. Complementi bibliografici ai "Fondamenti per la geometria sulle varietà algebriche: seconda memoria." Ann. Mat. Pura Appl. (4) **33**, 1–3 (1952).

Cf. same Ann. (4) **32**, 1–81 (1951); these Rev. **13**, 979.

Baldassarri, Mario. Sugl'insiemi di gruppi di punti generati da serie razionali. Rend. Sem. Mat. Univ. Padova **21**, 124–135 (1952).

The author considers aggregates of ∞^3 sets of points in the plane, generated by ∞^2 rational series of freedom one, and obtains conditions for such an aggregate to be linearly rational with respect to the series (i.e., that the sets can be mapped on an S_3 in such a way that the ∞^2 series are mapped by the lines of a star). He goes on to investigate the condition that a plane ∞^3 involution of order n should be the projection of a similar involution in S_3, showing that, when $n>2$, this reduces to the condition of linear rationality for the ∞^3 sets of $n-1$ points residual, in the involution, to the various points of the plane. *J. A. Todd.*

Baldassarri, Mario. I sistemi algebrici di spazi e l'insieme dei loro spazi totali. Rend. Sem. Mat. Univ. Padova **21**, 171–197 (1952).

A t-space S_t ($t>h$) is said to be a total space of a linear complex L, given by the equation $\sum a_{i_0\cdots i_h} p^{i_0\cdots i_h}=0$, of h-spaces in S_n if every S_h in S_t belongs to L. The t-space with Plücker coordinates $(\pi^{i_0\cdots i_t})$ is a total space of L if and only if $\sum a_{i_0\cdots i_h}\pi^{i_0\cdots i_h j_1\cdots j_{t-h}}=0$, for all choices of $j_1, \cdots, j_{t-h}$. To determine the dimension of the system of total t-spaces of L it is necessary only to determine the number of independent conditions of this form. In the first section of the paper this problem is solved in the case $t=h+1$, and it is shown that if L possesses τ independent total hyperplanes, the totality of total t-spaces is the intersection of $h+2-\tau$ independent linear complexes of t-spaces. In the second section a formula is found for the dimension of the system of t-spaces which are total for q independent general linear complexes of h-spaces, and in the third section the results are extended to some non-linear complexes of h-spaces. In a number of places it is asserted that certain loci are product spaces; the reviewer thinks that it can be asserted only that they are fibre bundles; but since only the dimensions of the constructs are in question, the validity of the conclusions is not affected. *W. V. D. Hodge.*

Turri, Tullio. Sopra sostituzioni unimodular involutorie relative a tabelle di periodi di integrali abeliani reali. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 7–10 (1952).

Se ω è la matrice di Riemann dei periodi degli integrali semplici di prima specie sopra una varietà algebrica reale, la matrice ω è equivalente alla sua complessa coniugata $\bar{\omega}$, sicchè $\bar{\omega}=\rho\omega A$, con A unimodulare e ρ matrice complessa non degenere. Scegliendo opportunamente gli integrali sopra la varietà si può anzi supporre che la ρ sia la matrice identica, onde sarà $\bar{\omega}=\omega A$. Tutto ciò essendo noto, l'A., riferendosi alle curve algebriche reali e supponendo che la ω abbia una delle forme canoniche, assegnate da Weichold-Klein rispettivamente per il caso delle curve ortosimmetriche, diasimmetriche e per il caso delle curve prive di circuiti reali, assegna esplicitamente la forma della matrice A.

F. Conforto (Roma).

Conforto, Fabio. Una proposizione sulle matrici quasi abeliane. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 335–345 (1950).

L'A., proseguendo le sue ricerche per una teoria aritmetica delle funzioni quasi abeliane nel senso di Severi, stabilisce il seguente lemma preliminare a successivi sviluppi: Sia ω una matrice quasi abeliana, cioè formata coi $\pi'(<2\pi)$ periodi di un corpo di funzioni quasi abeliane di π variabili; se α, C sono matrici quadrate a determinante non nullo, rispettivamente d'ordini π, π' e costituite da numeri complessi e da numeri razionali, riesce quasi abeliana anche la matrice $\alpha\omega C$. Mentre nel caso abeliano ($\pi'=2\pi$) il lemma è di immediata dimostrazione, non è così nel caso attuale in cui manca finora una caratterizzazione delle matrici quasi abeliane del tipo di quella, valida per le matrici di Riemann, basata sulla nozione di matrice principale.

E. Martinelli (Genova).

Abellanas, Pedro. Orientation of algebraic varieties. Revista Mat. Hisp.-Amer. (4) **12**, 79–101 (1952). (Spanish. English summary)

The present paper gives an arithmetical definition for the orientation of an algebraic irreducible variety V of the affine space over an orderable field of constants, k. It is proved that, if V contains some simple k-points, i.e., some points having their coordinates in k, then V has two opposite orientations. Considering two oriented subvarieties of V of complementary dimensions, a sign is then attributed to each of their isolated intersections, at a simple k-point of V which is also simple for both subvarieties. Next five lemmas on determinants are stated (the second and fifth of which, however, are not correct). Then an irreducible variety V is considered in an affine space E_n over a field $k^*=k(w)$, where k is orderable and k^* is an algebraic extension of rank ρ normal over k. An orientation of V is defined by means of an orientation of the variety image of V in $E_{n\rho}$, and it is shown that the orientation is unique or there are two orientations of V according as the norms of all the elements of k^* over k have or have not the same sign; in the first case, the algebraic number of intersections of two subvarieties of V of complementary dimensions equals the arithmetical number of intersections. *B. Segre* (Rome).

Weil, André. On Picard varieties. Amer. J. Math. **74**, 865–894 (1952).

In a recent paper [same J. **74**, 1–22 (1952); these Rev. **13**, 680] J. Igusa proved the existence of a duality between the Picard variety and the Albanese variety of a non-singular projective variety over the complex field. By defining a structure of complex vector space A' on the dual space A_0' of the underlying real vector space A_0 of a complex vector space A (A' being "antidual" to A), and by passage to the quotients by conjugate discrete subgroups of A and A', Weil defines the dual torus T' of a complex analytic torus T; each one is the character group of the fundamental group of the other; when T is an abelian variety, T' is also one. Igusa's results about algebraic varieties are extended to a "Hodge manifold" V, i.e., a complex analytic variety with a Kähler metric whose associated differential form Ω of degree 2 is cohomologous to an integral cocycle: existence of a complex analytic mapping F of V into the abelian variety T determined by the periods of the differential of the first kind on V, and of an isomorphism of G_a/G_l (G_a: group of analytic divisors on V which are homologous to 0 mod integers; G_l: group of divisors of meromorphic functions on V) onto the dual abelian variety T' of T. By use of theta-functions on the product $T\times T'$, it is proved that there exists on V an algebraic family (S) of divisors in G_a parametrized by T' and containing one and only one representative of every class in G_a/G_l. The main result of this paper is that the canonical mapping c of G_a onto T' induces an analytic mapping into T' of every analytic family $(F)\subset G_a$ of divisors on V; this result is conjectured to extend to the case of an algebroid family (F); the fact that the vector space $L(X)$ of meromorphic functions f on V such that the divisor (f) is greater than a given divisor $-X$ comes as a lemma, and is elegantly proved by using H. Cartan's results on ideals of holomorphic functions, the compactness of bounded sets of holomorphic functions, and the finite dimensionality of locally compact normed vector spaces. Lastly the author defines, for every compact connected Kähler manifold V of dimension n, a sequence T_i ($0\leqq i\leqq n-1$) of complex toruses, such that T_i and T_{n-i-1} are dual toruses, and such that T_0 and T_{n-1} are the above defined T and T'; when V is a Hodge manifold, these toruses are abelian varieties; under an additional natural condition of homological nature one can define a homomorphism c_i into T_i of the group $G_h^{(0)}$ of the i-dimensional analytic cycles on V which are homologous to 0 mod integers; but (as pointed

out by W. V. D. Hodge) this mapping is not onto in general, and this seems to exclude the hope that the study of its kernel might provide a good definition of linear equivalence for cycles which are not divisors. *P. Samuel.*

Chow, Wei-Liang. On Picard varieties. Amer. J. Math. **74**, 895–909 (1952).

Several results of the preceding paper are given alternative proofs in the case of an algebraic variety V. Using, as did Igusa, a "general curve" on V, Weil's duality between complex toruses (which is described here with coordinates), and the Chow-coordinates of cycles, the author proves simultaneously the theorems of Igusa and Weil's "main result" on the "analyticity" of the canonical mapping of G_a onto T' [see the preceding review]. Let $|C|$ be a suitable $(n-1)$-dimensional linear system of curves fibering V^n, and $C(u)$ a generic element of this system over a suitable field of definition K of V and $|C|$; the Jacobian variety $J(u)$ of $C(u)$ is defined over $K(u)$ according to a previous (not yet published) result of the author; the Albanese variety T of V is isomorphic to a quotient abelian variety $J(u)/J_1(u)$, and the Picard variety T' is isomorphic to an abelian subvariety of $J(u)$ ($J(u)$ being its own dual); using these results one constructs a representative system of divisors in G_a, parametrized by T', but this parametrization is only almost everywhere one-to-one. The discussion of the generic Jacobian variety $J(u)$ shows also that T' is isomorphic with the largest abelian subvariety of $J(u)$ defined over K; this is the starting point of a forthcoming purely algebraic construction of Picard varieties for algebraic varieties over an arbitrary ground field. *P. Samuel* (Ithaca, N. Y.).

Differential Geometry

Winger, R. M. A parametric treatment of polar tangent curves. Univ. Washington Publ. Math. **3**, 99–108 (1952).

A study of the curves whose polar equation is of the form $\rho = a \tan(p/q)\theta + k$, p and q being coprime integers and a, k real constants. *J. A. Todd* (Cambridge, England).

Ničе, V. Les surfaces strophoïdales du 3e ordre. Acad. Serbe Sci. Publ. Inst. Math. **4**, 113–120 (1952).

La surface générale du 3ième ordre peut être définie comme le produit d'une gerbe de droites de sommet S et d'une gerbe de quadriques en correspondance homographique. L'auteur envisage le cas où la gerbe de quadriques est constituée par les quadriques passant par une conique donnée C et deux points donnés A, B de l'espace, et rattache à ce mode particulier de définition de la surface du 3ième ordre un certain nombre de propriétés géométriques de la surface. Il suppose ensuite que la conique C soit l'ombilicale, et étudie en détail les nombreuses propriétés géométriques résultant de cette particularisation ainsi que des différentes dispositions possibles des points S, A, B, propriétés qui justifient la dénomination de surfaces strophoïdales du 3ième ordre que l'auteur leur a donnée. *P. Vincensini.*

Ničе, Vilko. Les surfaces strophoïdales du 3e ordre. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 97–112 (1952). (Serbo-Croatian. French summary)

Serbo-Croation version of the paper reviewed above.

Löbell, Frank. Natürliche Geometrie der Kurvenkongruenzen. Math. Z. **56**, 208–218 (1952).

If a congruence of curves is given, then at each point of space there is a uniquely determined unit vector e tangent to the directed curve of the congruence through this point. If through such a point we consider an orthogonal trajectory, with unit tangent a to the curves of the congruence, the two vectors a and e determine a trihedral of orthogonal unit vectors. Considerations analogous to the use of the moving trihedral in classical differential geometry lead to fundamental equations describing the behavior of the congruence. In case the congruence is a normal congruence the coefficients in the fundamental equations reduce to familiar quantities for the orthogonal surfaces of the congruence. *S. B. Jackson* (College Park, Md.).

Sakellariou, Nilos. Ueber Strahlensysteme deren abwickelbaren Flächen eine Fläche unter geodätischen Linien und ihren geodätischen Parallelen schneiden. Bull. Soc. Math. Grèce **26**, 69–74 (1952). (Greek summary)

The author obtains differential equations whose solution determines the rectilinear congruence whose developable surfaces intersect a given surface in its geodesic lines and geodesic parallels. *A. Fialkow* (Brooklyn, N. Y.).

Mishra, R. S. Five families of ruled surfaces through a line of a rectilinear congruence. III. J. Indian Math. Soc. (N.S.) **16**, 55–62 (1952).

On sait que Kummer a mis à la base de l'étude des congruences rectilignes deux certaines formes quadratiques des différentielles des variables u^α ($\alpha = 1, 2$) fixant les différents rayons: $f \equiv G_{\alpha\beta} du^\alpha du^\beta$, $\theta = \mu_{\alpha\beta} du^\alpha du^\beta$, et que Sannia a opportunément remplacé la deuxième forme de Kummer θ par une autre $\phi = \xi_{\alpha\beta} du^\alpha du^\beta$. Différents auteurs [Ogura, Hayashi, Ram Behari, Mishra] ont étudié les cinq familles de surfaces réglées d'une congruence, obtenues en annulant les formes différentielles fournies par l'annulation du résultat de l'opération Jacobienne $J(f, \phi)$, effectuée à partir de f et ϕ, et répetée autant que possible. L'auteur a fait une étude analogue pour les deux formes de Kummer f et θ, et, dans la note actuelle, il complète les résultats obtenus sur ce sujet en reprenant la question pour les formes θ et ϕ. Il note que les surfaces $J(\theta, \phi) = 0$ sont celles pour lesquelles le rapport entre la distance du point central au point correspondant de la surface de référence et le paramètre de distribution est constant. Il étudie ensuite les surfaces $J(\theta, J(\theta, \phi)) = 0$, et prouve qu'elles coïncident avec les développables de la congruence si la congruence est normale ou bien si la surface de référence est la surface moyenne. Puis il montre que, sous l'une ou l'autre de ces deux mêmes conditions, les surfaces d'équation $J(\phi, J(\theta, \phi)) = 0$ ont leurs lignes de striction sur la surface de référence. Un rapprochement avec des résultats antérieurs l'amène à cette conclusion que pour trois formes données f, θ, ϕ l'opération Jacobienne, répetée avec ces formes prises deux à deux, fournit en général dix surfaces distinctes, ces surfaces se réduisant à cinq dans les deux cas où la surface de référence est la surface moyenne et où la congruence est normale. L'auteur termine son étude par l'établissement de quelques propriétés relatives aux plans centraux des représentations sphériques des surfaces de distribution ou des surfaces principales d'une congruence rectiligne. *P. Vincensini* (Marseille).

Šul'man, T. A. Asymptotic transformations of triply conjugate systems of surfaces. Doklady Akad. Nauk SSSR (N.S.) **85**, 501–504 (1952). (Russian)

By a triply conjugate system of surfaces R is understood a triple system of surfaces such that the lines of intersection of one of these surfaces with the other two form a net R. Since one surface is an asymptotic transform of another if they are focal surfaces of a congruence W, the author is concerned with transformations of R which carry conjugate nets into conjugate nets and transform the corresponding surfaces asymptotically. This projective problem and the results are analogous to those of Bianchi on triply orthogonal systems of surfaces. *M. S. Knebelman* (Pullman, Wash.).

Klingenberg, Wilhelm. Über die 2-dimensionalen Flächen im 4-dimensionalen projektiven Raum. Arch. Math. **3**, 154–162 (1952).

In un recente lavoro [Math. Z. **55**, 321–345 (1952); questi Rev. **14**, 207], l'A. aveva trattato il problema di determinare, per un punto P (regolare) di una F_p in P_n (spazio reale proiettivo a n dimensioni) un riferimento locale intrinseco, e aveva dimostrato che, se lo $(m+1)$-esimo spazio osculatore a F_p in P coincide con P_n, è possibile determinare, mediante certe condizioni di "apolarità generalizzata", il riferimento (che dipenderà in generale dalla calotta di ordine $2m+1$ di centro P) in modo univoco, non appena si scelga nello S_p tangente ad F_p in P uno S_{p-1} non passante per P, N_0. Nel caso poi di $m=1$, era possibile determinare anche N_0, e il riferimento perfettamente individuato dipendeva in questo caso dalla calotta del 4° ordine di centro P. Nel presente lavoro, l'A. applica i risultati or ora citati alle F_2 in S_4 (per le quali appunto $m=1$), dotate di un doppio sistema coniugato, che si suppone reale. In questa ipotesi, le formule del lavoro sopra ricordato permettono di assegnare (nn. 1 e 2) il riferimento locale, e uno sviluppo canonico nell'intorno di P, quando si prendano come linee coordinate u, v, su F_2 quelle del doppio sistema coniugato. Nel n. 3 si caratterizza geometricamente il sistema di riferimento trovato; uno dei vertici è naturalmente il punto P, mentre due degli altri coincidono con i trasformati di Laplace di P in entrambi i versi; dei rimanenti due vertici si dànno diverse caratterizzazioni, una delle quali, per esempio, si ottiene considerando la polarità rispetto a certe quadriche aventi contatto del 3° ordine con F_2 in P. Nel n. 4, infine, si dànno alcuni risultati concernenti classi particolari di superficie, e si confronta il riferimento trovato con quelli introdotti da altri AA., che abbisognavano però di alcune ipotesi restrittive, non necessarie per l'A. [Il revisore osserva che all'A. pare sfuggito il fatto che Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **5**, 84–90, 143–149 (1927)] aveva determinato un riferimento locale intrinseco per una superficie possedente un doppio sistema coniugato, indipendentemente dalla dimensione dello spazio ambiente; anche il riferimento del Bompiani dipende dalla calotta del 4° ordine di P; esso esige però che i due invarianti di Laplace del doppio sistema coniugato siano differenti.]

V. Dalla Volta (Roma).

Graiff, Franca. Sulla possibilità di costruire parallelogrammi chiusi in alcune varietà a torsione. Boll. Un. Mat. Ital. (3) **7**, 132–135 (1952).

It is shown that if two elements d_1x^h and d_2x^h in the same point are displaced pseudoparallel along each other, one of them with the parameters L^h_{ij}, the other with L^h_{ji}, one gets a closed quadrangle, which the author calls a parallelogram.

J. Haantjes (Leiden).

Schneidt, Max. Über die endlichen Gleichungen einer Fläche, deren sphärisches Bild gegeben ist. Arch. Math. **3**, 70–75 (1952).

In an earlier paper [Arch. Math. **2**, 367–374 (1950); these Rev. **13**, 71] the author derived a partial differential equation for all surfaces with a linear element $Edu^2+2Fdudv+Gdv^2$ with the exception of those with $E=G=0$. There he gives another system of equations including all cases. He considers the element in the form $(\alpha du+\gamma dv)^2+(\beta du+\delta dv)^2$ and proves that

$$\left(-m\frac{\gamma_u}{\gamma}+\alpha\frac{\partial^2 \log F}{\partial u\partial v}\right)du+\left(m\frac{\alpha_v}{\alpha}+\gamma\frac{\partial^2 \log F}{\partial u\partial v}\right)dv$$

is a total differential if $ds^2=2\alpha\gamma dudv$. The new system is connected with the spherical representation in §2, and in §3 applications are given for several kinds of surfaces. The representation considered is not real and in §4 it is shown for the case of surfaces of constant curvature that equally simple results may be obtained by means of a real representation.

J. A. Schouten (Epe).

Muracchini, Luigi. Le varietà V_3 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. II. Rivista Mat. Univ. Parma **3**, 75–89 (1952).

This paper completes the work started in part I [same Rivista **2**, 435–462 (1951); these Rev. **14**, 79]. The V_3 satisfying d Laplace equations ($5\leqq d\leqq 10$) whose tangent S_3 fill up a variety of dimension 9 are enumerated; there are fourteen classes of varieties in the list. *J. A. Todd.*

Muracchini, Luigi. Alcune proprietà in grande delle trasformazioni puntuali fra spazi. Boll. Un. Mat. Ital. (3) **7**, 123–131 (1952).

Si ottengono alcune proprietà in grande delle trasformazioni puntuali fra spazi, relative alle direzioni e giaciture caratteristiche. (Author's summary.) *J. A. Todd.*

Vaona, Guido. Sulle trasformazioni puntuali fra piani aventi due reti asintotiche di curve caratteristiche corrispondenti. Boll. Un. Mat. Ital. (3) **7**, 148–154 (1952).

The author's principal result is as follows. The point-point transformations between planes such that two corresponding nets of characteristic curves are asymptotic nets are those, and only those, which can be obtained by projecting, from two fixed centres, pairs of corresponding points of two non-developable surfaces which are projectively applicable (or, in particular, homographic). *J. A. Todd.*

Clement, Mary Dean. A criterion for determining the space of immersion of a variety of arbitrary dimensionality. Proc. Amer. Math. Soc. **3**, 715–722 (1952).

La condition nécessaire et suffisante pour qu'une courbe analytique définie dans un espace projectif S_n à n dimensions appartienne à un espace d'immersion S_k ($k<n$) s'exprime habituellement par une équation différentielle linéaire et homogène d'ordre $k+1$.

Etant donné une variété analytique V_m, à n dimensions, définie dans S_n par le vecteur $x=x(u_1, u_2, \cdots, u_m)$, l'auteur considère une suite de matrices dont la première se réduit à la colonne des $n+1$ coordonnées homogènes de x, dont la matrice d'indice h se déduit de la précédente, d'indice $h-1$, en lui adjoignant les colonnes dérivées partielles d'ordre h de la colonne x. L'auteur étudie quelques propriétés des ces

matrices et montre que, pour que V_n admette un espace d'immersion d'ordre k, il faut et il suffit que la matrice d'indice $k-m+2$ de la famille précédente soit de rang $k+1$. La condition peut également s'exprimer par un système complètement intégrable d'équations aux dérivées partielles. *M. Decuyper* (Lille).

Angelitch, Tatomir. Verallgemeinerung des Begriffs des Darboux'schen Vektors für den Raum von Riemann. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 147–158 (1952). (Serbo-Croatian. German summary)

With a curve in E_3 one may associate the skew-symmetric tensor whose elements $\omega_{12}=-\omega_{21}$ and $\omega_{23}=-\omega_{32}$ are the two curvatures, respectively, and whose remaining elements vanish. The author studies an analogous tensor for curves in an n-dimensional Riemannian space. *W. Feller.*

Mishra, R. S. Sur certaines courbes appartenant à un sous-espace d'un espace riemannien. Bull. Sci. Math. (2) **76**, 77–84 (1952).

The author generalizes the definition of union curves and hyperasymptotic curves of a hypersurface to apply to curves in any subspace of a Riemann space. By straightforward tensor calculations, the differential equations of these curves are derived and a number of their geometric properties are obtained. An example of the latter is the following theorem: If the normal curvature vector of a subspace for the direction of a union curve is zero, then the union curve is a geodesic of the subspace. *A. Fialkow* (Brooklyn, N. Y.).

Sen, R. N. On an algebraic system generated by a single element and its applicatiän in Riemannian geometry. III. Bull. Calcutta Math. Soc. **43**, 77–94 (1951).

This paper continues the detailed study of the algebraic system begun in an earlier paper [same Bull. **42**, 1–13 (1950); these Rev. **12**, 205]. *A. Fialkow.*

Nožička, F. La connexion et la normale de l'hypersurface dans l'espace riemannien du point de vue de la géométrie affine. Czechoslovak Math. J. **1(76)**, 17–28 (1951) =Čehoslovack. Mat. Ž. **1(76)**, 19–31 (1951).

In a previous paper [Časopis Pěst. Mat. Fys. **75**, 179–209 (1950); these Rev. **13**, 72], the author studied the affine normal of a hypersurface in an affinely connected space and the affine connection induced on the hypersurface by this affine normal. In the paper under review, the author shows that if the enveloping space is a Riemannian space, then the affine normal and the induced connection defined by the author in the previous paper coincide with ordinary ones. In the last half of the paper, the author studies in detail the cases in which the rank of the second fundamental tensor is less than $n-1$. *K. Yano* (Tokyo).

Eisenhart, Luther P. Generalized Riemann spaces. II. Proc. Nat. Acad. Sci. U. S. A. **38**, 505–508 (1952).

Starting from the connexions $\{{}^{h}_{ij}\}$ and Γ^{h}_{ij} defined in part I [Proc. Nat. Acad. Sci. U. S. A. **37**, 311–315 (1951); these Rev. **13**, 279] and the curvature affinors belonging to them, five other affinors of valence four are derived algebraically from these affinors, the torsion affinor $\Gamma^{h}_{[ij]}$, and its covariant derivative. There are a great number of algebraical relations. By contraction of one of these affinors the Einstein tensor is refound. There are no physical interpretations. *J. A. Schouten* (Epe).

Adati, Tyûzi. On subprojective spaces. II. Tôhoku Math. J. (2) **3**, 330–342 (1951).

[For part I see same J. **3**, 159–173 (1951); these Rev. **14**, 86.] Rachevsky [Trudy Sem. Vektor. Tenzor. Analizu **1**, 126–142 (1933)] obtained three necessary and sufficient conditions that a Riemann space V_n be subprojective. A considerable portion of the present paper is concerned with Rachevsky's third condition which may be written as

$$T_{\lambda\mu}=\rho g_{\lambda\mu}+\rho_{,\lambda}\sigma_{,\mu} \tag{1}$$

where $T_{\lambda\mu}$ is defined by $T_{\lambda\mu}=(n-2)^{-1}[R_{\lambda\mu}-\frac{1}{2}(n-1)^{-1}Rg_{\lambda\mu}]$. The author considers the family of hypersurfaces $\eta(x^\lambda)=$ constant, assuming that (1) may be written as

$$T_{\lambda\mu}=\rho g_{\lambda\mu}+\kappa\eta_{,\lambda}\eta_{,\mu}. \tag{2}$$

He finds that in a V_n admitting a concircular vector field $\eta_{,\lambda}$, (2) is the necessary and sufficient condition that the hypersurfaces $\eta=$ constant are all Einstein spaces. Also, in this case, ρ, κ, and R must all be functions of η only if $n>3$. The effect of conformal transformations of V_n upon equation (2) is considered. A typical result is the following: In order that the form of (2) (with ρ, κ functions of η only) remain invariant under a conformal transformation $\bar{g}_{\mu\nu}=\sigma^2 g_{\mu\nu}$ it is necessary and sufficient that η^λ be a concircular vector field and σ be a function of η only. Other theorems concern themselves with the Ricci directions of a conformally flat Riemann space. *A. Fialkow* (Brooklyn, N. Y.).

Adati, Tyûzi. On subprojective spaces. III. Tôhoku Math. J. (2) **3**, 343–358 (1951).

The major portion of the paper deals with a subprojective space which admits a parallel vector field. In this case, it is shown that Rachevsky's first two conditions hold unchanged but that the third condition is (2) of the preceding review with ρ and κ constant. Here $\eta_{,\lambda}$ is a parallel vector field. The Christoffel symbols of the second kind may be written as $\{{}^{\lambda}_{\mu\nu}\}=\phi_\mu\delta^\lambda_\nu+\phi_\nu\delta^\lambda_\mu+\phi_{\mu\nu}\xi^\lambda$ where ξ^λ is a parallel vector field. Various theorems concerning a Riemann space which admits a parallel vector field are obtained of which the following is an example: Let V_n ($n>3$) be a Riemann space which admits a family of ∞^1 totally geodesic hypersurfaces whose normals are a parallel vector field. Then the tangential directions of these hypersurfaces are Ricci principal directions if and only if all the hypersurfaces are Einstein spaces. *A. Fialkow* (Brooklyn, N. Y.).

Adati, Tyûzi. On subprojective spaces. IV. Tensor (N.S.) **1**, 105–115 (1951).

The subpaths with respect to the vector field ξ^λ of an affinely connected space A_n have been defined by K. Yano [Proc. Imp. Acad. Tokyo **20**, 701–705 (1944); Math. Japonicae **1**, 51–59 (1948); these Rev. **7**, 331; **10**, 403]. He has also derived the changes of the coefficients of connection (called subprojective changes) which transform the system of subpaths into itself. This work leads naturally to the study of the space whose coefficients of connection may be written as

$$\Gamma^\lambda_{\mu\nu}=\phi_\mu\delta^\lambda_\nu+\phi_\nu\delta^\lambda_\mu+\phi_{\mu\nu}\xi^\lambda. \tag{3}$$

An infinitesimal transformation

$$\bar{x}^\lambda=x^\lambda+\epsilon\xi^\lambda \tag{4}$$

which leaves the subpaths invariant is called a subprojective collineation. It is proved that, in a Riemann space, the transformation (4), where ξ^λ is a torse-forming vector field, is a subprojective collineation if and only if ξ^λ is a concircular vector field. Also, an affinely connected space whose coeffi-

cients of connection are given by (3) is a subprojective space if (a) the vector field ξ^λ is torse-forming, (b) the transformation (4) is a subprojective collineation, and (c) ϕ_μ is a gradient vector. *A. Fialkow* (Brooklyn, N. Y.).

Adati, Tyûzi. On subprojective spaces. V. Tensor (N.S.) **1**, 116–129 (1951).

Results analogous to the last result of the preceding review are obtained. It is shown that the theorem is true either if condition (c) is replaced by further enumerated conditions on the vector ξ^λ or if condition (a) is replaced by (a′) the vector field ξ^λ is a parallel one and condition (c) by (c′) $\xi^\lambda\phi_\lambda\neq 0$. A detailed study of changes of the vector ϕ_λ in subprojective Riemann and affine spaces is made. *A. Fialkow* (Brooklyn, N. Y.).

Adati, Tyûzi. On subprojective spaces. VI. Tensor (N.S.) **1**, 130–136 (1951).

The effect of subprojective changes of affine connection upon the Riemann and Ricci curvature tensors is studied. It is also shown that a subprojective space admitting a torse-forming field ξ^λ transforms into a subprojective space under a subprojective change of affine connection with respect to the same vector field ξ^λ. By considering the transformation law of the projective curvature tensor $W^\lambda{}_{\mu\nu\omega}$ under a subprojective change of connection, necessary and sufficient conditions that $W^\lambda{}_{\mu\nu\omega}$ be unaltered are obtained. *A. Fialkow* (Brooklyn, N. Y.).

Egorov, I. P. A tensor characteristic of A_n of nonzero curvature with maximum mobility. Doklady Akad. Nauk SSSR (N.S.) **84**, 209–212 (1952). (Russian)

The author solves the problem of finding all spaces of affine connection A_n with nonzero curvature which admit the complete group of motions of maximum order, i.e., n^2. Further, the paper contains a tensor form of the necessary and sufficient condition that a given A_n with nonzero curvature be of maximum mobility. 1) A previous result of Egorov [same Doklady (N.S.) **80**, 709–712 (1950); these Rev. **14**, 208] shows that the following condition is necessary: A_n is a projectively-euclidean space. 2) A further necessary condition can be obtained from the integrability conditions of the equations of an infinitely small motion V^α and $V^\alpha{}_{,\beta}$: (a) A_n is an equiprojective space; (b) the Ricci tensor fulfills $R_{\alpha\beta}=(1-n)\epsilon\lambda_\alpha\lambda_\beta$, $\epsilon=\pm 1$; (c) $\lambda_{\alpha,\beta}=C\lambda_\alpha\lambda_\beta$, where C is a suitable constant and $\lambda_{\alpha,\beta}$ is the symmetrical part of a tensor which fulfills $\lambda_{\alpha,[\beta}\lambda_{\gamma]}=0$.

Further, the existence of A_n with nonzero curvature fulfilling (a), (b), (c) is investigated. The author obtains the following results: 1) Under certain initial conditions there exists a unique A_n of nonzero curvature fulfilling (a), (b), (c). 2) A given A_n of nonzero curvature has maximum mobility if and only if it fulfills (a), (b), (c), for a suitable constant c. *F. Vyčichlo* (Prague).

Egorov, I. P. Maximally mobile L_n with a semi-symmetric connection. Doklady Akad. Nauk SSSR (N.S.) **84**, 433–435 (1952). (Russian)

In previous papers [same Doklady (N.S.) **64**, 621–624 (1949); **73**, 265–267 (1950); these Rev. **10**, 739; **12**, 636] the author has shown that the maximum order of the group of motions of a space L_n with affine connection is n^2. Spaces L_n with this order of group of motions will be called maximally mobile spaces. The present paper is devoted to the discussion of necessary and sufficient conditions that a given L_n be maximally mobile. Further, the problem of finding all maximally mobile L_n in a given X_n is considered.

Let L_n be a maximally mobile space with a semi-symmetrical connection $\Lambda_{\beta\gamma}{}^\alpha$ and tensor of torsion $\Omega_{\beta\gamma}{}^\alpha$ which fulfills $\Omega_{\beta\gamma}{}^\alpha=\delta_\beta{}^\alpha\Omega_\gamma-\delta_\gamma{}^\alpha\Omega_\beta$, where Ω_α is a vector. Let $R_{\beta\gamma\delta}{}^\alpha$ be the components of the tensor of curvature of the connection $\Lambda_{(\beta\gamma)}{}^\alpha$. The integrability conditions of the equations of an infinitely small motion V^α may be written by means of the Lie derivation

$$D_L R_{\beta\gamma\delta}{}^\alpha=0,$$
$$D_L\Omega_{\beta\gamma}{}^\alpha=0.$$

The group of motions of L_n is a subgroup of the complete group of motions of A_n with the connection $\Lambda_{(\beta\gamma)}{}^\alpha$. Hence the following necessary conditions for L_n may be deduced: (a) A_n is equiprojective; (b) $R_{\alpha\beta}=(1-n)c_1\Omega_\alpha\Omega_\beta$; (c) $\Omega_{\alpha\beta}=c_2\Omega_\alpha\Omega_\beta$, where c_1, c_2 are invariants.

Using these results the author proves the following theorems. 1) Let A_n be a space with a semi-symmetrical connection, the torsion tensor of which is defined by the vector Ω_α. The space L_n is maximally mobile if and only if it fulfills the conditions (a), (b), (c) for suitable constants c_1 and c_2. 2) Under certain initial conditions there exists a unique L_n with a semi-symmetrical connection fulfilling (a), (b), (c), where c_1, c_2 are the suitable constants. 3) A maximally mobile L_n with a semi-symmetrical connection is symmetrical if and only if $\Omega_{\alpha\beta}=0$. 4) The group of motions of a maximally mobile L_n coincides with the complete group of motions of A_n if and only if A_n has a nonzero curvature.

In the case that L_n is a group-L_n, i.e., a space with a group of connections, the author proves: 5) A group-L_n with a semi-symmetrical connection is maximally mobile. 6) A group-L_n with a torsion is maximally mobile if and only if every pair of operators of the corresponding group represents a subgroup with the pair of elements *F. Vyčichlo.*

Tachibana, Syun-ichi. On a relation between two non-Euclidean connexions of Einstein spaces. Nat. Sci. Rep. Ochanomizu Univ. **2**, 31–33 (1951).

Let E_n be an Einstein space with positive definite metric tensor. It is known that one can construct, from the E_n, (1) a space C_n with normal conformal connection whose group of holonomy fixes a hypersphere A, and (2) a space P_n with normal projective connection whose group of holonomy fixes a hyperquadric B. The author points out that the equations which define the connection for the C_n can be easily rewritten so that they are of the same form as those for the connection of the P_n, and he gives a geometric explanation for this resemblance. The n-dimensional tangent space at each point of C_n is considered as an $(n+1)$-dimensional projective space. A hyperplane $\bar{P}_n$ is selected in this space in a simple way, and it is shown how the Möbius transformations which fix A induce projective transformations in $\bar{P}_n$ which fix a hyperquadric $\bar{B}$. The equation of $\bar{B}$ is of the same form as that of B. *A. Schwarts.*

Akivis, M. A. Invariant construction of the geometry of a hypersurface of a conformal space. Mat. Sbornik N.S. **31(73)**, 43–75 (1952). (Russian)

A partly expository paper on conformal differential geometry of a hypersurface. The first three sections are introductory and are devoted to multispherical coordinate representation and the study of the subgroups of the conformal group leaving the surface element invariant. In the next four sections is given an invariant construction of the theory by means of the stationary subgroups associated

with the hypersurface. By this means it is possible to introduce conformal tensors g_{ij}, a_{ij}, c_{ij} and $B_k=b_{ijk}a^{ij}$ which determine the hypersurface to within a conformal mapping. The remainder of the paper is devoted to some special problems dealing with the order of contact of certain cyclides with the hypersurface, particularly for the critical cases of the enveloping space being of 3 and of 4 dimensions.

M. S. Knebelman (Pullman, Wash.).

Izmaĭlov, V. D. On the invariant geometry of a surface of six-dimensional affine space. Doklady Akad. Nauk SSSR (N.S.) **85**, 477–480 (1952). (Russian)

The theory of surfaces X_2 in an affine space E_n of dimension n was treated by C. Burstin and W. Mayer for $n=4$ [Math. Z. **26**, 373–407 (1927)], by A. Kawaguchi [Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **17**, 43–68 (1928)] and by G. F. Laptev [Dissertation, Moscow, 1950], by making use of a tensor of second order on X_2. In the same way, the present author discusses the case $n=6$, introducing the tensor of second order $\gamma_{\alpha\beta}=\mu_{\alpha\beta}|\mu|^{-5/11}$, where $\mu_{\alpha\beta}=2^2\lambda_{[\alpha|[1|}\lambda_{1|\beta]\eta}$, $\lambda_{\alpha\beta}=\det|r_1\, r_2\, r_{11}\, r_{12}\, r_{22}\, r_{\alpha\beta}|$ and $|\mu|=\det|\mu_{\alpha\beta}|$, provided that X_2 is given in the vector form $r=r(x^1, x^2)$. Also, $\beta_{\alpha\beta}=\lambda_{\alpha\beta}|\mu|^{-2/11}$ is a tensor satisfying the apolarity relation $\beta_{\alpha\beta}\gamma^{\alpha\beta}=0$ and its geometrical meaning is given. On making use of the covariant derivatives by means of $\gamma_{\alpha\beta}$, the Frenet formulas in an equi-affine space E_6 are deduced; from these follow the complete system of invariants and the fundamental theorem. Under a general affine transformation, $\gamma_{\alpha\beta}$ is multiplied only by a constant and the theory will be obtained by a slight modification. The special case $|\mu|=0$ is also discussed.

A. Kawaguchi (Sapporo).

Norden, A. P. On the geometrical theory of finite deformations. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 53–61 (1950). (Russian)

The author asks for renewed interest in the work of J. Le Roux [Ann. Sci. Ecole Norm. Sup. (3) **28**, 523–579 (1911); **30**, 193–245 (1913)] on infinitesimally small and on finite deformations, and presents his results in the language of the tensor calculus. He introduces a system of curvilinear coordinates in space, u^i, $i, j, k, \cdots=1, 2, 3$; a deformation maps the point M with radius-vector $OM=r(u^1, u^2, u^3)$ on a point M' with $OM'=\bar{r}(u^1, u^2, u^3)$. Then the vectors

$$a=a^i r_i, \quad \bar{a}=a^i\bar{r}_i, \quad r_i=\partial_i r, \quad \bar{r}_i=\partial_i\bar{r}$$

are called corresponding with respect to the deformation. If $\partial_j r_i=G^k{}_{ij}r_k$, $\partial_j\bar{r}_i=\Gamma^k{}_{ij}\bar{r}_k$, then $A^k{}_{ij}=\Gamma^k{}_{ij}-G^k{}_{ij}$ is called the tensor of the affine deformation. To the G belongs the covariant differential operator D, to the Γ belongs Δ. When $A^k{}_{ij}=0$ the deformation is homogeneous: $\bar{r}=\phi r+c$, ϕ a constant affinor.

If $r_i=r_i{}^k\bar{r}_k$ and $D_k\bar{r}_j=\rho^l{}_{kj}\bar{r}_l$, then $A^s{}_{jk}$ can be written as $-r^s{}_m\rho^m{}_{jk}$. Since the curvature tensor with respect to the $G^k{}_{ij}$ and to the $\Gamma^k{}_{ij}$ is zero, the equivalent conditions hold

$$D_{[i}A^l{}_{j]k}-A^m{}_{k[i}A^l{}_{j]m}=0, \quad \nabla_{[i}A^l{}_{j]k}+A^m{}_{k[i}A^l{}_{j]m}=0.$$

Let $ds^2=g_{ij}du^i du^j$, $d\sigma^2=\gamma_{ij}du^i du^j$ in the spaces (r) and (ρ). Then

$$A^i{}_{jk}=\tfrac{1}{2}\bar{\gamma}^{im}(D_j\gamma_{km}+D_k\gamma_{jm}-D_m\gamma_{jk}), \quad \bar{\gamma}^{im}\gamma_{mj}=\delta_j{}^i,$$

and

$$A^m{}_{mk}=\partial_k \lg(\epsilon_{123}/e_{123})=(\partial_k\theta)/(1+\theta),$$

where

$$L=e_{ijk}a^i b^j c^k, \quad \Lambda=\epsilon_{ijk}a^i b^j c^k$$

are the volumes of corresponding parallelepipeds, constructed with the aid of the alternating trivectors e_{ijk}, ϵ_{ijk}, and $1+\theta=\lim \Lambda/L=\epsilon_{123}/e_{123}$.

The theory is then applied to curves, their tangent unit vectors t^k, τ^k and curvature vectors kn^i, $\varkappa\nu^i$. For the unit binormal vectors b^k, β^k the formula is obtained:

$$\varkappa\beta_i=\frac{1+\theta}{(1+e)^3}(kb_i+e_{ipq}A^q{}_{ms}t^p t^m t^s), \quad e=e_{123}.$$

The following derived tensors of Le Roux are discussed

$$p_{ij}=-\tfrac{1}{2}e^{mn}{}_i A_{mnj}$$
$$\tau_{ij}=p_{(ij)} \text{ (the torsion tensor)}$$
$$\delta_{ijk}=e_{ijs}p^s{}_k=\tfrac{1}{2}(A_{jik}-A_{ijk})$$
$$\varphi^i=-\tfrac{1}{2}e^{imn}p_{mn}$$
$$c_{ijk}=\tfrac{1}{4}(A_{ijk}+A_{ikj}+A_{kij}),$$

where the raising and lowering of indices is performed by means of g_{ij} and $e_{ijk}=(r_i r_j r_k)$. Then

$$A_{ijk}=c_{ijk}+\tfrac{1}{2}e_{ijs}\tau^s{}_k+\tfrac{2}{3}(g_{ik}\varphi_j+g_{ij}\varphi_k-2g_{jk}\varphi_i).$$

The tensor c_{ijk} can be written in the form

$$c_{ijk}=B_{ijk}+\tfrac{1}{4}(c_i g_{jk}+c_j g_{ki}+c_k g_{ij}), \quad c_i=c^m{}_{mi},$$

where the tensor B_{ijk} is symmetrical and satisfies the condition $B^m{}_{mi}=0$. This allows us to express the tensor $A^m{}_{jk}$ by means of two symmetrical tensors B_{ijk} and τ_{ij}, for which $B^m{}_{mi}=0$ and $\tau^k{}_k=0$, and two vectors φ_i, c_i. The number of components is $B_{ijk}\to 7$, $\tau_{ij}\to 5$, $\varphi_i\to 3$, $c_i\to 3$, $A_{ijk}\to 18$.

Several special cases are discussed, especially the case of infinitesimal deformations, given by

$$\bar{r}=r+\bar{u}=r+u^k r_k,$$

where the quadratic terms in the u^k and their derivatives may be neglected. Then $A^k{}_{ij}=D_j D_i u^k$, $2\epsilon_{ij}=\tau_{ij}-g_{ij}=2D_{(i}u_{j)}$ and $p_{kl}=-\tfrac{1}{2}D_k w_l$, where $w_l=e^{mn}{}_l D_m u_n$ is the vector of the infinitesimal rotation. We also find $\tau_{kl}=-\tfrac{1}{2}D_{(k}w_{l)}$ and $\varphi_i=-\tfrac{1}{2}e_i{}^{mn}D_m$.

D. J. Struik (Cambridge, Mass.).

Liber, A. E. On the theory of surfaces in a centro-affine (vector) space. Doklady Akad. Nauk SSSR (N.S.) **85**, 37–40 (1952). (Russian)

The author remarks that an n-dimensional vector space B_n can be considered as a geometrical n-space with a fundamental group, the group of its automorphisms; this space is isomorphic with a central-affine n-space E_n. Then the concepts of an m-surface, m-plane, etc., can be carried over into vector space, and with it the notion of rigging (Einspannung). This allows him to set up conditions which determine an n-surface in B_n but for automorphisms. The study connects with results of V. V. Vagner [Trudy Sem. Vektor. Tenzor. Analizu **7**, 65–166 (1949); **8**, 11–72, 197–272 (1950); these Rev. **13**, 777, 281, 778] and K. H. Weise [Math. Z. **44**, 161–184 (1938)].

D. J. Struik.

Takizawa, Seizi. On generalized spaces which admit given holonomy groups. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **26**, 199–209 (1951).

Let R^n be an n-dimensional generalized manifold in the sense of Cartan, with generalized tangent spaces, for which a connection is defined. The transformations of the tangent space at a point A of R^n into itself, induced by displacements around closed curves through A form the holonomy group $\mathfrak{H}$ of the space. A class $\mathfrak{F}$ of frames R_A at A is called a "Ricci family" of $\mathfrak{H}$ if $\mathfrak{F}$ is invariant under $\mathfrak{H}$, i.e., if $\mathfrak{H}\mathfrak{F}=\mathfrak{F}$. Let each "coset" of $\mathfrak{F}$ (i.e., a minimal subset of $\mathfrak{F}$ that is a Ricci family) be characterized by parameters y^i. The basic theorem in the investigations is: "If R^n admits a

holonomy group $\mathfrak{H}$, we can take R_A so that Pfaffian forms of connection may satisfy an arbitrary Ricci relation of $\mathfrak{H}$. In this case the invariants y^j are functions of $x^{\theta''}$ (the coordinates) "but they depend on the choice of the coset to which R_A belongs." The paper further gives equations to be satisfied by the Pfaffian forms of the connection, and it pays special attention to the case when $\mathfrak{H}$ is intransitive or imprimitive. *A. Nijenhuis* (Princeton, N. J.).

***Nijenhuis, Albert. Theory of the geometric object.** Thesis, University of Amsterdam, Amsterdam, 1952. xvi+238 pp. (English. Dutch summary) $2.00 (distributed by the author).

The work is divided into three chapters. The first chapter is based on two fundamental concepts: "transformation element" and "allowable coordinate systems". A transformation element $(\overset{a}{\xi}{}^r, \overset{a}{f}{}^r)$ is a point $\overset{a}{\xi}{}^r$ and a set of n functions $\overset{a}{f}{}^r(\xi)$ analytic in some neighborhood of $\overset{a}{\xi}{}^r$ with non-vanishing functional determinant at $\overset{a}{\xi}{}^r$,

$$\overset{a}{f}{}^r(\xi^\lambda)=\overset{a^{-1}}{\xi}{}^r+\sum_{p=1}^{\infty}\frac{1}{p!}\overset{a}{a}{}^r{}_{\lambda_p\cdots\lambda_1}(\xi^{\lambda_p}-\overset{a}{\xi}{}^{\lambda_p})\cdots(\xi^{\lambda_1}-\overset{a}{\xi}{}^{\lambda_1}).$$

Here $\overset{a^{-1}}{\xi}{}^r$ is the point into which $\overset{a}{\xi}{}^r$ is transformed by $\overset{a}{f}{}^r$. A set of coordinate systems is "allowable" if at every point the transition between every pair of coordinate systems (covering the point) leads to a transformation element. The transition from one coordinate system (defined for a point of a geometrical manifold) to every other system is determined in some neighborhood of the point by a transformation element and vice versa. Instead of using the Veblen-Whitehead pseudo-group of transformations [The foundations of differential geometry, Cambridge, 1932] the author uses the groupoid of transformation elements, where the word groupoid is meant in the sense of Brandt [Math. Ann. **96**, 360–366 (1926)]. This groupoid is simply isomorphic with a groupoid Γ_n which is a product of the groupoid of the set of pairs $\langle\overset{a}{\xi}{}^r, \overset{a^{-1}}{\xi}{}^r\rangle$ (with a simple composition rule) and the group which is the set of elements $\overset{a}{a}{}^r{}_{\lambda_p\cdots\lambda_1}$, $p=1, 2, \cdots$ (satisfying some necessary inequalities). The differential geometry in the small is the invariant theory of the aforementioned groupoid. Thus for instance the (micro-) geometric objects as well as the macrogeometric objects may be defined by means of this groupoid. It turns out that the transformations of components of a geometric object constitute a groupoid that is a representation of Γ_n. It is worthwhile to mention that in the Schouten-Haantjes standard theory of objects [Proc. London Math. Soc. (2) **42**, 356–376 (1937)] the transformations of components of a geometric object do not constitute a representation of the pseudo-group of point transformations.

The second chapter deals mainly with the deformation theory of fields of geometric objects (which may even be functionals of some other geometric object) and is based on the concept of "group-germ". This concept is defined in an abstract way as follows. Let $\mathfrak{G}$ be a group of elements $a, b, \cdots$ with the unit element e. Let $\mathfrak{G}^*$ be a subset of $\mathfrak{G}$ such that 1) for every $a \in \mathfrak{G}$ there exists a finite number of elements $a_1, \cdots, a_k \in \mathfrak{G}^*$ such that $a=a_1a_2\cdots a_k$, 2) $e \in \mathfrak{G}^*$, 3) if $a \in \mathfrak{G}^*$, then $a^{-1} \in \mathfrak{G}^*$. If these conditions are satisfied, then $\mathfrak{G}^*$ is called a group-germ. Most of the so-called "groups of transformations" in differential geometry in the small are continuous group-germs of partial transformations (with respect to the operation of local multiplication). Defining the "dragging along" in the usual way the author obtains Lie-derivatives (even in non-holonomic coordinates) of geometrical objects. After deriving various general properties of the Lie-derivative (as, for instance, the commutation formula and the Lie-Taylor series of Kosambi which enable the author to deal also with finite deformation) the author applies these results on different topics among which the most interesting seems to be the investigation of the group of holonomy. The final result may be stated as follows: The (ξ) domain of the generators $C_\lambda{}^r$ of the group of holonomy of an L_n is the (ξ) domain of the set of affinors

$$R_{\omega\mu\lambda}{}^r, \quad \nabla_{a_p\cdots a_1}R_{\omega\mu\lambda}{}^r, \quad p=1, 2, \cdots,$$

where R is the curvature tensor of the L_n.

While the second chapter deals with components of objects, the third chapter deals with their transition induced by the transformation element which characterizes the change of coordinates. Among many topics considered by the author we mention only the formulation of criteria that an object be a differential geometric object.

While in this review only some instances of the author's investigations were briefly mentioned, the following table of contents might give more information. Chapter I, "The geometric object": 1) Introduction; 2) Transformation elements in arithmetic manifold; 3) Further examination of the groupoid of transformation elements; 4) The geometric manifold; 5) The concept of a geometric object; 6) The definition of geometric objects; 7) Types of geometric representations; 8) Notations—The kernel-index principle; 9) Examples; 10) The Fundamental Theorem for special objects; 11) A possible generalisation of the theory; 12) Some analytic operations. Chapter II, "Deformations of geometric objects": 1) Introduction; 2) Point transformations in a geometric manifold; 3) The process of dragging along; 4) Infinitesimal transformations—The Lie-derivative; 5) The Lie-derivative in explicit form; 6) Lie-derivation in anholonomic coordinates; 7) Finite deformations—The Lie-Taylor series; 8) Lie-derivation with respect to different fields; 9) Continuous groups of transformations; 10) The linear geometric object of finite representation class; 11) A commutation formula; 12) A set of identities for the Lagrangian derivative; 13) The generators of the group of holonomy; 14) Some remarks on pseudo-parallel displacements. Chapter III, "The geometric representation": 1) Introduction; 2) Notations; 3) Transitivity, primitivity and equivalence of geometric representations; 4) Some properties of group $\mathfrak{G}_n$; 5) The simple geometric representation. *V. Hlavatý* (Bloomington, Ind.).

Aussem, M. V. Geometry of the double integral in three-dimensional space. Doklady Akad. Nauk SSSR (N.S.) **85**, 253–255 (1952). (Russian)

Applying the invariant method of Laptev [same Doklady (N.S.) **78**, 197–200 (1951); these Rev. **13**, 280] and Vasil'ev [ibid. **79**, 5–7 (1951); these Rev. **13**, 159], the connection and invariants in the three-dimensional Cartan space are obtained, on making use of the theory of differential forms and notations of E. Cartan [Ann. Sci. Ecole Norm. Sup. (3) **22**, 219–308 (1905)]. That is, the invariant relation with respect to the double integral $\int\int[\omega^1\omega^2]$ is given by $\omega_1{}^1+\omega_2{}^2=a_{Ii}\omega^I+b^i\omega_i{}^3$ $(I=1, 2, 3;\ i=1, 2)$ from which the metric tensor g^{ij} and the connection $\omega_K{}^I$ are determined, operating the external differentiations successively. If $\omega^3=0$, $\omega_i{}^3=a_{ij}\omega^j$ for a surface, then b^i gives its transversal and $g^{ij}a_{ij}=0$ for an extremal surface. Finally there is a remark

that this discussion can be easily generalized to the n-dimensional Cartan space. *A. Kawaguchi* (Sapporo).

Moór, Arthur. Über oskulierende Punkträume von affinzusammenhängenden Linienelementmannigfaltigkeiten. Ann. of Math. (2) **56**, 397–403 (1952).

Following a study of Finsler spaces by Varga [Monatsh. Math. Phys. **50**, 165–175 (1941); these Rev. **5**, 218], the author defines an osculating point space B for an affinely connected line-element space L^* as follows. B is a domain in L^* such that one and only one path of a family of paths in L^* passes through every point of B. Then the absolute differential and the curvature tensor of L^* coincide with those of B respectively, and the paths of L^* and of B osculate each other in the second order when they have a line-element in common, just as in the metric case (i.e., Finsler space). *A. Kawaguchi* (Sapporo).

Fulton, C. M. Weierstrass tensors. Amer. Math. Monthly **59**, 544–547 (1952).

NUMERICAL AND GRAPHICAL METHODS

***Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leqq x \leqq 1$.** National Bureau of Standards Applied Mathematics Series, No. 25. U. S. Government Printing Office, Washington, D. C., 1952. x+60 pp. $.40.

Reissue of tables which first appeared in 1948 as no. 1 in the same series [see these Rev. **10**, 150].

Siegel, K. M., Crispin, J. W., Kleinman, R. E., and Hunter, H. E. The zeros of $P_{n_i}'(x_0)$ of non-integral degree. J. Math. Physics **31**, 170–179 (1952).

The problem is to determine, for a given θ, values of n such that $P_n'(\cos\theta)=0$, and also to determine, for these n_i,

$$\int_{\cos\theta}^{1} P_{n_i}'(x)dx. \tag{1}$$

Tables of such values were published by Carrus and Treuenfels [same J. **29**, 282–299 (1951); these Rev. **12**, 638] and criticized by Siegel, Brown, Hunter, Alperin, and Quillen [Engrg. Res. Inst., Univ. of Michigan, Rep. no. **UMM-82**, 1951; these Rev. **13**, 387] and Y. L. Luke [Math. Tables and Other Aids to Computation **5**, 152–153 (1951)]. In the present paper the authors describe a comparatively quick method of computation which is sufficiently accurate for engineering work. The method is carried out in numerical detail for $\theta=165°$ in which case $P_n(\cos\theta)$ and $-P_n'(\cos\theta)/\sin\theta$ were tabulated for $n=-.5(.5)20.5$ and also approximations for the first 19 roots n_i with the corresponding integrals (1) were computed. *A. Erdélyi* (Pasadena, Calif.).

Hammersley, J. M. Lagrangian integration coefficients for distance functions taken over right circular cylinders. J. Math. Physics **31**, 139–150 (1952).

Eight-place tables are given for the purpose of facilitating the calculation of integrals of the form $\int\int t(r)r^{-2}dvdv'$, where the integral is taken over all volume elements dv and dv' in a circular cylinder, r is the distance between these elements, and $t(r)$ is an arbitrary function. In a previous paper [Proc. Roy. Soc. London. Ser. A. **210**, 98–110 (1951); these Rev. **13**, 690] the author has reduced the problem of evaluating this sixfold integral to that of calculating simple integrals with respect to a variable s whose integrands are products of $t(s)$ and certain fixed functions $g(s)$ or $h(s)$. The tables in this paper give products of the Lagrange 11-point or 5-point coefficients times values of g and h, so that the product integrals can be approximated as a sum of products of values from the tables times values of $t(s)$. *P. W. Ketchum.*

Guest, P. G. Curve-fitting by the method of grouping. Australian J. Sci. Research. Ser. A. **5**, 238–257 (1952).

This follows a previous article [Ann. Math. Statistics **22**, 537–548 (1951); these Rev. **13**, 481], except that here the author considers methods appropriate when the x's are not equally spaced. In this case, the usual orthogonal polynomial tables cannot be used; hence, a fairly efficient grouping method is desirable. The y's are grouped so as to maximize the efficiencies of the estimated regression coefficients when the x's are equally spaced. Some investigations have shown that these efficiencies are not reduced materially for unequally spaced x's, unless many observations are clustered at one end of the range. Methods of grouping are presented for polynomials through the fourth degree and number of observations $n=10(1)100$. The efficiencies for equally spaced x's are given for large n and specifically for $n=20$, 36, 48, 60, 75, 80, 95, 100. An example is presented for $n=16$, with unequally spaced x's. Approximate formulas for the standard errors of the regression coefficients are also presented. *R. L. Anderson* (Raleigh, N. C.).

Michalup, Erich. Recent developments in interpolation formulae. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **14**, no. 45, 14–38 (1951). (Spanish summary)

Ostrowski, A. M. On the rounding off of difference tables for linear interpolation. Math. Tables and Other Aids to Computation **6**, 212–214 (1952).

Sackmann, Louis A. Mise en équation des résultats d'expériences. Lois empiriques linéarisées. C. R. Acad. Sci. Paris **235**, 689–691 (1952).

Sackmann, Louis A. Mise en équation des résultats d'expériences. Nouvelles formules d'utilisation. C. R. Acad. Sci. Paris **235**, 783–784 (1952).

Pflanz, Erwin. Über die Beschleunigung der Konvergenz langsam konvergenter unendlicher Reihen. Arch. Math. **3**, 24–30 (1952).

When a convergent series $\sum a_n$ has terms satisfying appropriate conditions, it is possible to calculate the terms of a new series $\sum b_n$ which converges more rapidly to the same value. Suppose, for example, $\sum a_n=s$ and

$$a_n/a_{n-1}=1-\alpha/n+\gamma_{n-1}/n$$

where $\alpha>1$ and $\gamma_n\to 0$ as $n\to\infty$. One of the series treated is the series $\sum b_n'$ for which

$$b_0'=a_0\left[1+\frac{\alpha}{\alpha-1}\frac{a_1}{a_0-a_1}\right]$$

and

$$b_n'=a_n\left[1+\frac{\alpha}{\alpha-1}\left(\frac{a_{n+1}}{a_n-a_{n+1}}-\frac{a_{n-1}}{a_{n-1}-a_n}\right)\right]$$

when $n>0$. For the case of the series $\sum a_n=\sum_{n=0}^{\infty}(n+1)^{-4/3}$, which converges to $2.6123753487\cdots$, $a_0+\cdots+a_8=1.963713$ while $b_0'+\cdots+b_8'=2.612674\cdots$. Iteration of this same

procedure gives a new series $\sum b_n''$ for which

$$b_0''+\cdots+b_7''=2.61237\cdots.$$

Other procedures, some appropriate for alternating series, and pertinent references are given. *R. P. Agnew.*

Spencer, Roy C., and Parke, Nathan Grier, III. A matrix treatment of the approximation of power series using orthogonal polynomials including applications. Antenna Laboratory, Electronics Research Division, Air Force Cambridge Research Center, Cambridge, Mass., 1952. 78 pp.

Suppose the Taylor series $\sum_0^\infty a_n h_n x^n$ represents $f(x)$ for $|x|\leqq 1$, where the h_n are scale factors. Let $f_N(x)=\sum_0^N a_n h_n x^n$. There is a unique expansion $f_N(x)=\sum_0^N k_n\phi_n(x)$, where the ϕ_n are given polynomials, orthogonal on an interval I with respect to a weight factor $w(x)$. For $M<N$, let $f_{NM}(x)=\sum_0^M k_n\phi_n(x)=\sum_0^M a_{NMn}h_n x^n$ be a truncation of the above expansion. This $f_{NM}(x)$ is the best least-squares approximation of degree M to $f_N(x)$ in the interval I, with the weight factor $w(x)$. When $|f(x)-f_N(x)|$ is small, $f_{NM}(x)$ is usually a practically useful, nearly uniform approximation to $f(x)$ on I. The purpose of the paper is to give detailed directions for computing the a_{NMn} from the a_n by use of auxiliary matrices which are tabulated explicitly (for $N\leqq 13$) for the Tchebycheff, Legendre, Hermite, and Laguerre classical polynomials. There are several examples illustrating this "economization" procedure, which was inspired by Lanczos [J. Math. Physics **17**, 123–199 (1938)]. Changing the interval I, and use of even and odd polynomials are discussed. *G. E. Forsythe* (Los Angeles, Calif.).

Stein, Marvin L. Gradient methods in the solution of systems of linear equations. J. Research Nat. Bur. Standards **48**, 407–413 (1952).

Let a matrix A have m rows and n columns, while b, x are vectors of m, n rows. Let H be an $m\times m$ positive Hermitian matrix. Let $r=r(x)=b-Ax$, and $f(x)=r^*Hr$ (* means conjugate transpose). Given A, b, one seeks x minimizing $f(x)$. If $m=n$ and $|A|\neq 0$, such an x solves $Ax=b$. Following unpublished work with M. R. Hestenes, extending ideas of Cauchy, Temple, and Kantorovič, the author considers a general gradient method for minimizing $f(x)$. Let x_0 be arbitrary. For $i=0, 1, 2, \cdots$ let $x_{i+1}=x_i+\beta_i\gamma_i A^*Hr(x_i)$, where the real number γ_i minimizes $f(x_i+\gamma A^*Hr(x_i))$ as a function of the complex parameter γ. If all $|\beta_i-1|\leqq\delta<1$ (β_i can be complex), then $f(x_i)\downarrow \min f(x)$.

There is a detailed report on one problem $Ax=b$ with $m=n=6$, run (for $i=0, \cdots, 30$) on an IBM Card-Programmed Calculator with $\beta_i=\beta$ taking various real values. For $1\leqq\beta\leqq 1.9$, $f(x_{i+1})/f(x_i)\uparrow$, eventually exceeding .97, so that $f(x_i)$ decreased only slowly. For $.1\leqq\beta\leqq .95$, $f(x_{i+1})/f(x_i)$ varied irregularly; on the average, $f(x_i)$ decreased most rapidly for $\beta=.9$. These "almost optimum" methods $\beta<1$ are "self-accelerating," in contrast to those for $\beta\geqq 1$, and thus may have considerable value for solving $Ax=b$ on an automatic machine when there is no room to store a special acceleration routine. *G. E. Forsythe.*

Maslov, P. G. Method of solution of systems of homogeneous linear equations in the computation of the vibrations of polyatomic molecules. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 276–283 (1952). (Russian)

It is desired to find a scalar λ and a vector r such that (*) $(G\lambda-A)r=0$, where (the reviewer presumes) the nth order matrices G, A are symmetric and G is definite. As in an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **71**, 867–870 (1950); these Rev. **12**, 152], the author deals with an iterative process; here he sets the first components of all vectors $r_{(s)}$ ($s=0, 1, 2, \cdots$) equal to 1. If $\lambda_{(s)}$, $r_{(s)}$ are known, one can (i) solve the last $n-1$ equations (*) for $r_{(s+1)}$, and (ii) using $r_{(s+1)}$, solve the first equation (*) for $\lambda_{(s+1)}$. To speed the convergence the author proposes replacing step (ii) by a step (iii) too complicated to reproduce. It looks as though the algorithm (i), (iii) may be equivalent to solving the nonlinear system (*) for the n unknowns λ, r by Newton's method, but the reviewer did not verify this. There are formulas for calculating the partial derivatives of λ with respect to the elements of G. The vibration of the ethane molecule is used in a brief numerical example.

G. E. Forsythe (Los Angeles, Calif.).

Tietz, Horst. Beweis der Konvergenz eines Verfahrens von W. Bartky zur Berechnung von bestimmten Integralen. J. Reine Angew. Math. **189**, 246–249 (1952).

The author discusses Bartky's method [Rev. Modern Physics **10**, 264–269 (1938)] for the evaluation of integrals between the limits 0 and $\frac{1}{2}\pi$ of the form $\int F(R)R^{-1}dt$ where $R^2=m^2\cos^2 t+n^2\sin^2 t$, which are generalizations of the complete elliptic integrals. In this method m, n, R, and $F(R)$ are replaced by sequences m_i, n_i, R_i, and $F_i(R_i)$ which are defined recursively, the m_i and n_i by arithmetic and geometric means, respectively, and R_i by a kind of Newton approximation to the square root of n_i^2. Then, according to Bartky, for any continuous $F(R)$, the integral can be approximated by replacing the integrand by the constant $F_i(R_i)/R_i$. The present author proves the convergence of the method as i tends to infinity under the assumption that $F(R)$ is of bounded variation and then deduces Bartky's statement with the aid of Weierstrass' theorem on the approximation of continuous functions by polynomials.

P. W. Ketchum (Urbana, Ill.).

Collatz, L. Einschliessungssätze bei Iteration und Relaxation. Z. Angew. Math. Mech. **32**, 76–84 (1952).

Suppose that a system of n equations $f_k(x_1, \cdots, x_n)=0$, $k=1, \cdots, n$, in the n real unknowns $x_1, \cdots, x_n$, is solved numerically by an iteration process and that at one step of this process it is known that all the correction terms for the unknowns have the same sign. The question is raised as to whether, in succeeding steps in the iteration, the alteration of the correction terms will proceed in the same sense, and if it is possible in this way to give upper and lower bounds for the solution. In general the answer to this question is no, but the author answers it in the affirmative for some systems which occur in the approximate solution of boundary value problems for certain ordinary and partial differential equations. Particular attention is paid to the iteration methods of total and successive steps and to the relaxation method.

J. B. Diaz (College Park, Md.).

Söchting, Fritz. Zur Berechnung von Eigenschwingungszahlen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1951**, 132–135 (1951).

The present note contains several remarks concerning the computation of the eigenfrequencies ω^2, which satisfy the differential equation

$$\omega^2 v=F\left(x, y, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}, \cdots\right),$$

where v is the displacement. The author concludes that if an approximation v_1 differs little in absolute value from the actual displacement v, then the corresponding approximation to ω^2 computed from v_1 by an energy principle will also differ little from ω^2. These considerations are related to an earlier paper by the author [Federhofer-Girkmann Festschrift, Beiträge zur angewandten Mechanik, Deuticke, Wien, 1950, pp. 365–375]. *J. B. Diaz.*

***Mühly, Hans J. Die genäherte Berechnung von Eigenwerten elastischer Schwingungen anisotroper Körper.** Ergebnisse der exakten Naturwissenschaften, Band 24, pp. 402–442. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1951.

This is a detailed review of the subject matter and it would be impossible to discuss the contents minutely in a brief account, but an enumeration of some of the section headings should furnish an idea of the contents. Part I is entitled "Foundations", and contains four sections: 1) The fundamental equations of the mathematical theory of elasticity in Cartesian coordinates; 2) Transformation to other coordinate systems; 3) Formulation of the eigenvalue problems; 4) Eigenvalues, eigenfunctions and completeness theorems. Part II is entitled "Direct approximation methods" and contains, among others, sections on Ritz' method for obtaining upper bounds for eigenvalues, the iteration method and Grammel's method, and the computation of lower bounds for eigenvalues. (There is no mention of the method of A. Weinstein [Etude des spectres des équations aux dérivées partielles . . . , Gauthier-Villars, Paris, 1937] which yields lower bounds for the eigenvalues.) Part III, entitled Perturbation theory, contains, among others, sections on bounds for the eigenvalues, and perturbation by varying the domain. *J. B. Diaz.*

Rapoport, I. M. A new method of approximate integration of ordinary linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **85**, 955–958 (1952). (Russian)

The author considers an nth order system of linear differential equations expressed in the form $X'(t)=A(t)X(t)$ in which $A(t)$ is an nth order known matrix and $X(t)$ is an nth order matrix with unknown elements. This equation is solved by a process of matrix iteration which, with suitable limitations, is shown to converge uniformly in some interval of the t-axis. The method is illustrated for the case $n=2$ by a simple example. *W. E. Milne* (Corvallis, Ore.).

Luzin, N. N. On the method of approximate integration of academician S. A. Čaplygin. Uspehi Matem. Nauk (N.S.) **6**, no. 6(46), 3–27 (1951). (Russian)

In 1919 Čaplygin published his "Foundations of a new method of approximate integration of differential equations" [cf. Čaplygin, Collected works, vol. 1, Moscow-Leningrad, 1948, pp. 348–368]. In the present paper Luzin states that there is great need for a profound and unified exposition of Čaplygin's ideas, and that he is undertaking to give a report on the present status of the theory and to compare the present status with the original conception. There follows a recapitulation of Čaplygin's original treatment of the problem. *W. E. Milne* (Corvallis, Ore.).

Richter, W. Graphische Lösung von gewöhnlichen Differentialgleichungen mit nomographischen Hilfsmitteln. Z. Angew. Math. Mech. **32**, 120–129 (1952). (German. English, French and Russian summaries)

The direction field defined by a first-order differential equation of appropriate form is represented graphically by (a) two nomograms and a fixed curve, (b) four nomograms, or (c) other similar combinations. Starting with a given point $P(x, y)$, the values of x and y determine a point in each nomogram. In (b) these four points are connected in pairs by two lines intersecting in a point P' which when joined to P gives the desired direction at P. In (a) the point P' is determined by the intersection of one index line and the fixed curve. Certain special higher order differential equations can also be solved. Several examples are worked out. *P. W. Ketchum* (Urbana, Ill.).

Hovanskiĭ, A. N. Numerical solution of a nonlinear differential equation. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **1**, 95–98 (1948). (Russian)

The author corrects slight errors in the coefficients of s^4 and s^6 in Nyström's power series solution for

$$\varphi''(s)=\sin\varphi(s)-1;\quad \varphi(\pm\tfrac{1}{2})=0.$$

W. E. Milne (Corvallis, Ore.).

Hovanskiĭ, A. N. Approximate solution of a transcendental equation from the theory of stability of a rectangular plate. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 129–131 (1950). (Russian)

The author examines an equation given by Timoshenko [Theory of plates and shells, McGraw-Hill, New York and London, 1940, p. 316, equation (i)] and obtains a solution claimed to be more accurate than that of Timoshenko. *W. E. Milne* (Corvallis, Ore.).

Schäfer, Manfred. Eine graphische Richtungsfeldkonstruktion für den Phasenplan nichtlinearer freier Schwingungen. Z. Angew. Math. Mech. **32**, 284–286 (1952).

Èl'sgol'c, L. Ė. On approximate integration of differential equations with retarded argument. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 771–772 (1951). (Russian)

The paper discusses the numerical integration of equations of the type $|dx(t)/dt|=f(t, x(t), x(t-\tau))$, where $x(t)$ is given as $\varphi(t)$ for $t_0\leqq t\leqq t_0+\tau$. Various methods are recommended, all of which combine the method of successive approximations for differential equations on certain subintervals with other numerical integration techniques. *E. A. Coddington* (Los Angeles, Calif.).

Gibbons, J. J., and Schrag, R. L. A method of solving the wave equation in a region of rapidly varying complex refractive index. J. Appl. Phys. **23**, 1139–1142 (1952).

Beard, M., and Pearcey, T. An electronic computer. J. Sci. Instruments **29**, 305–311 (1952).

Wolf, J. Jay. The Office of Naval Research Relay Computer. Math. Tables and Other Aids to Computation **6**, 207–212 (1 plate) (1952).

Booth, A. D. The physical realization of an electronic digital computor. Electronic Engrg. **24**, 442–445 (1952).

Raymond, F. Analogies électriques et mécaniques. Rev. Gén. Electricité **61**, 465–475 (1952).

Beevers, C. A., and Lipson, H. The use of Fourier strips for calculating structure factors. Acta Cryst. **5**, 673–675 (1952).

Whiteley, T. B., and Alldredge, L. R. A photomechanical wave analyser for Fourier analysis of transient waveforms. J. Sci. Instruments **29**, 358–362 (1952).

Carter, D. S. An electrical method for determining journal-bearing characteristics. J. Appl. Mech. 19, 114–118 (1952).

Steffensen, J. F. Inequalities in Makeham-graduated tables. Skand. Aktuarietidskr. 35, 36–47 (1952).

It is shown that if the force of mortality at age x is given by

$$\mu_x = A + Bc^{\gamma x},$$

the relation expressing the continuous annuity $\bar{a}_x$ in terms of the annuity payable annually in advance $\ddot{a}_x$ may be written

$$\bar{a}_x = \begin{cases} \ddot{a}_x - \frac{1}{2} - \frac{1}{12}\theta(\mu_x+\delta), & A+\delta \geqq \frac{1}{4}\gamma, \quad 0<\theta<1, \\ \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x+\delta) + R, & 0<A+\delta \leqq \gamma, \end{cases}$$

where

$$|R| < (\gamma^3/384)(11+9\zeta+\zeta^2+\zeta^3), \quad \zeta = (\mu_x - A)/\gamma$$

and δ is the force of interest. It is also proved that $1/\bar{a}_x - \mu_x - \delta$ increases steadily with x towards its limit γ. *H. L. Seal.*

MECHANICS

Čerkudinov, S. A., and Speranskiĭ, N. V. On the synthesis of plane link mechanisms with stops. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov, no. 43, 5–12 (1951). (Russian)

In a previous paper by the first author, inaccessible to this reviewer [The method of multiple interpolation in mechanism design, same Trudy 11, no. 40, 5–48 (1950)] the first three sections deal with four-bar curves tangent to a circular arc at both its endpoints. Graphical methods are given for the determination of such curves, given the positions of the linkage crank corresponding to the endpoints of the circular arc. The present paper translates the graphical method into an analytical one. A numerical example is given.

A. W. Wundheiler (Chicago, Ill.).

Dobrovol'skiĭ, V. V. On the mechanism of generation of a curve accompanying a given one. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov, no. 43, 50–54 (1951). (Russian)

Let $OACB$ be a hinged parallelogram, and $OAC'B$ the corresponding antiparallelogram. The curves traced by C and C', when O is fixed, are called "associated". If the trace of C is algebraic of equation $F(x, y)=0$, the author puts $x = a\cos\alpha + b\cos\beta$, $y = a\sin\alpha + b\sin\beta$ where $OA=a$, $OB=b$, $\angle XOA = \alpha$, $\angle XOB = \beta$. This transforms the equation into $\sum A_{mn}\cos(m\alpha \pm n\beta + \delta) = 0$ where m and n are integers, and $\delta = 0$ or $\pm\pi$. The author shows how to mechanize the last relation by adding to $OACB$: as many planetary transmissions as there are terms in the last summation; six times as many bars; and a linkage for straight-line guidance. He also explains the procedure for deriving the equation of the associated cruve. *A. W. Wundheiler* (Chicago, Ill.).

Pöschl, Th. Über Hauptschwingungen mit endlichen Schwingweiten. Ing. Arch. 20, 189–194 (1952).

The author considers some properties of vibrations of mechanical systems, particularly vibrations such that all particles of the system pass through their equilibrium positions simultaneously and reach extreme positions simultaneously. The discussion is carried on chiefly in terms of a few specific examples, and it contains little or nothing that is genuinely new. *L. A. MacColl* (New York, N. Y.).

Stojanović, Rastko D. Differential equations of the movement of a rigid body in tensorial form. Bull. Soc. Math. Phys. Serbie 4, nos. 1–2, 43–49 (1952). (Serbo-Croatian. English summary)

The equations are those appearing in Whittaker, "A treatise on the analytical dynamics . . . , example on p. 44 [4th ed., Cambridge, 1927] in different notation and more explicit form. The author seems to regard the use of summation convention as sufficient for tensorial form. The separate terms of the equations are not tensors.

A. W. Wundheiler (Chicago, Ill.).

Stumpff, K. Eine einfache symmetrische Ableitung der Lagrangeschen partikulären Lösungen des Dreikörperproblems. Astr. Nachr. 280, 91–93 (1951).

In questa nota l'autore ritrova le soluzioni di Lagrange del problema dei tre corpi in una maniera estremamente semplice ed elegante, domandandosi civè sotto quali condizioni ognuno dei tre punti materiali descrive una conica sotto l'attrazione newtoniana degli altri due.

G. Lampariello (Messina).

Iglesias Garrido, Tomás. A generalization of the particular solutions of Lagrange of the three body problem. Revista Mat. Hisp.-Amer. (4) 11, 266–276 (1951). (Spanish)

L'autore generalizza le soluzioni di Lagrange del problema dei tre corpi al caso in cui tre punti materiali si attraggono reciprocamente in ragione diretta delle masse, la legge di dipendenza dell'attrazione dalla distanza r essendo espressa da una funzione $f(r)$, positiva e continua per $r \geqq 0$, eventualmente nulla per $r=0$ ed $r=\infty$. Alcune considerazioni valgono anche per il caso meno semplice del sistema di n punti materiali. *G. Lampariello* (Messina).

Bucerius, H. Determinierung der klassischen Mechanik durch zeitliche Randwerte. Astr. Nachr. 280, 233–244 (1952).

It is proposed that the problems of classical mechanics be reformulated as boundary value problems rather than as initial value problems. This is justified on the grounds that initial positions and velocities are difficult to observe accurately; the lack of success of the Laplace method of orbit determination, based on such initial conditions, is cited as an example. The motion of a single particle in a force field is studied extensively and various forms of natural boundary conditions are presented. The differential equations are reduced to integral equations by use of Green's functions; the corresponding linear differential operator is chosen either as the inertia term or as this term plus part of the applied force, when that part is linear. A consideration of boundary conditions at the ends of an infinite integral leads to singular integral equations. Many examples are given and interpreted physically. *W. Kaplan* (Ann Arbor, Mich.).

Haag, Rudolf. Der kanonische Formalismus in entarteten Fällen. Z. Angew. Math. Mech. 32, 197–202 (1952). (German. English, French and Russian summaries)

Consider the equations of motion of a dynamical system in the variational form known as Hamilton's principle,

$$(1) \qquad \delta \int L(q_\alpha, \dot{q}_\alpha)\,dt = 0 \quad (\alpha = 1, 2, \cdots, n),$$

where L is the Lagrangian. If we form the Hamiltonian $H = -L + p_\alpha \dot{q}_\alpha$, where the variables $\dot{q}_\alpha$ are to be eliminated

by means of the equations

$$p_\alpha = \partial L/\partial \dot{q}_\alpha, \tag{2}$$

the original equations (1) may be replaced by the equivalent system known as Hamiltonian or canonical equations of dynamics. In the paper under review the "canonical formalism" is developed for the case where the functional determinant $|\partial^2 L/\partial q_\alpha \partial \dot{q}_\alpha|$ of the system (2) vanishes. This leads to a system with superfluous variables p_α, q_α, i.e., relations $\Phi_i(p_\alpha, q_\alpha)=0$ $(i=1, 2, \cdots, r)$ hold, where $n-r$ is the rank of the system (2). The reduction of the number of the independent variables and the meaning of a Poisson bracket relation $(A, B)=C$ between two functions A, B of p_α, q_α are discussed. It is shown that in order that such a relation in p_α, q_α be meaningful, the functions A and B must satisfy the conditions $(A, \Phi_i)=(B, \Phi_i)=0$. *E. Leimanis.*

Galissot, François. Sur une méthode universelle de formation des équations du mouvement des systèmes matériels. C. R. Acad. Sci. Paris **234**, 2148–2150 (1952).

On peut associé à un système matériel S' une forme extérieure de Cartan

$$\Omega = \int k_{ij}(dv^i \wedge dx^j)\delta m - \left(\int k_{ij}v^i dv^j \delta m\right) \wedge dt + \left(\int dx_i \delta F^i\right) \wedge dt$$

dans laquelle m désigne une mesure de masse, F une mesure de force (intégration de Radon), et k_{ij} le symbole de Kronecker. Dans le cas S est holonome les équations du mouvement sont les équations associées à Ω au sens de Cartan [Les systèmes différentiels extérieurs . . . , Hermann, Paris, 1945; ces Rev. **7**, 520]. L'auteur indique comment on peut obtenir les équations du mouvement dans le cas ou le système S' est astreint à des liaisons non holonome.

J. Haantjes (Leiden).

Gallissot, François. Application des formes extérieures du 2e ordre à la dynamique Newtonienne et relativiste. Ann. Inst. Fourier Grenoble **3** (1951), 277–285 (1952).

L'auteur montre que pour un système matériel continu à 6 paramètres (q^i, p_i) de position et vitesse on peut lui associer une forme extérieure de Cartan bilinéaire $\Omega(\delta q^i, \delta p_i; dq^i, dp_i, dt)$. Ω est nulle pour tout choix des fonctions δq^i qui satisfont aux équations de Killing et de δp_i arbitraire si les d correspondent aux équations différentiels du mouvement. En relativité restreinte on peut associer à un point matériel une forme extérieure de Cartan

$$\omega = m_0 c^2 k_{ij}[du^i \wedge dx^j - u^i du^j \wedge ds] + k_{ij}F^i dx^j \wedge ds$$

avec $k_{ij}=0$ $(i \neq j)$, $k_{11}=k_{22}=k_{33}=-k_{44}=-1$. Les u^i sont 4 paramètres liés par la relation $k_{ij}u^i u^j = 1$. Les équations du mouvement sont les équations associées à ω. [Cf. l'oeuvre analysé ci-dessus.] *J. Haantjes* (Leiden).

***Finzi, Bruno. Meccanica razionale. Volume primo. Teorie introduttive, cinematica, statica.** Seconda edizione. Nicola Zanichelli Editore, Bologna, 1950. xii +343 pp. 2000 lire.

Second edition of an extensive, carefully composed and well-written textbook. Vector and tensor calculus, not purely mathematically treated and with many applications, are given in an introduction (pp. 1–95). Kinematics follow (pp. 97–219) with its subdivisions: displacement, motion, velocity, acceleration (all for points, systems, and rigid bodies) and relativistic kinematics. The last part gives statics, for the point, the rigid body, and deformable systems, and ends with a chapter on virtual displacements. The book is mathematically satisfying and gives, moreover, many physical examples; it is an excellent textbook on the subject, both scientifically and pedagogically.

O. Bottema (Delft).

***Finzi, Bruno. Meccanica razionale. Volume secondo. Dinamica.** Seconda edizione. Nicola Zanichelli Editore, Bologna, 1950. xi+425 pp. 2000 lire.

The second volume has the same quality as the first. To give an impression of its contents we remark that it deals, e.g., with the general theorems of dynamics, the principles of thermodynamics, the mechanics of relative motion, relativistic dynamics, units and dimension of physical quantities, Poinsot's motion of a rigid body, the "symbolic" equations of dynamics, d'Alembert's principle, stability of equilibrium, small vibrations, Maggi's equations for anholonomic systems, canonical equations, variation principles, dynamics of continuous systems, impact, statistical mechanics. *O. Bottema* (Delft).

Biggeri, Carlos. New general methods in rational ballistics. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 199, Serie segunda, 19, Revista 4 (1951), 314–351 (1952). (Spanish)

The author proposes to give "several distinct general methods which he has obtained" for solution of the first and second ballistic problems on the whole of the trajectory, whatever the resistance function adopted. For brevity of exposition he refers specifically to the principal problem of exterior ballistics but points out that the methods are applicable to total differential equations generally. Making use of Lipschitz conditions he proceeds recursively and claims to show that series solutions in t (the time) result for each of the ballistic variables, convergent for all real positive values of t. No quantitative results are exhibited. The author lists some 170 authors of studies in ballistics followed by "etc., etc., etc.,". These all, including Popoff, are said to have failed to recognize the solubility here shown forth.

A. A. Bennett (Providence, R. I.).

Eckel, Karl. Beziehungen zwischen den Störungskoeffizienten der ballistischen Störungsrechnung. Z. Angew. Math. Physik **3**, 309–312 (1952).

As pointed out by P. Stanke [Wehrtechn. Monatsh. **42**, 560–569 (1938); **43**, 35–45, 63–69 (1939)], the first order perturbation effects in the ballistic variables, u, z, x, t, (z measured vertically upward), are linear functions of the first order changes in initial data (angle of elevation, muzzle velocity, logarithm of the ballistic coefficient, and logarithm of the absolute temperature) with perturbation coefficients. The author shows that on developing the perturbations at the origin in terms of t, dividing by t, and passing to a limit, convenient explicit linear equations for the perturbation coefficients are obtained. *A. A. Bennett.*

Paleček, E. M. Approximate integration of the equations of exterior ballistics by the method of S. A. Kazakov for vertical parameters. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 505–510 (1952). (Russian)

Kazakov's method [same journal **9**, 129–138 (1945); these Rev. **7**, 139] is extended to the case where either the ordinate of the center of gravity of an artillery shell or the vertical component of the velocity of the center of gravity is taken as the independent variable [cf. Paleček, ibid. **15**, 83–96 (1951); these Rev. **12**, 760]. *E. Leimanis.*

Hydrodynamics, Aerodynamics, Acoustics

Serrin, James. Two hydrodynamic comparison theorems. J. Rational Mech. Anal. 1, 563–572 (1952).

En précisant certains raisonnements de Lavrentieff, l'auteur a récemment développé une méthode élégante pour l'étude de l'unicité de la solution du problème du sillage plan [Amer. J. Math. 74, 492–506 (1952); ces Rev. 13, 877]; cette méthode repose sur l'établissement de deux théorèmes sur la comparaison des vitesses en un point de la ligne libre, pour les mouvements du type considéré. L'extension de ces résultats aux écoulements méridiens a été inaugurée par Gilbarg [J. Rational Mech. Anal. 1, 309–320 (1952); ces Rev. 13, 877], qui a donné pour ces écoulements l'analogue du premier théorème de Serrin [loc. cit.]. Dans le présent travail, l'auteur établit, pour les mouvements de ce type, l'analogue du second théorème de son oeuvre cité. La démonstration utilise le fait que la fonction de courant d'un champ méridien vérifie une équation du type elliptique, dont les solutions jouissent de la propriété de ne pas avoir d'extrêmum à l'intérieur du domaine de définition. Deux corollaires de ce théorème fournissent une estimation quantitative de la vitesse en un point d'inflexion de la ligne libre.

R. Gerber (Toulon).

Kravtchenko, Julien. Note sur les solutions approchées du problème déterminé des sillages. Ann. Inst. Fourier Grenoble 3 (1951), 287–299 (1952).

This is an account of the method of I. M. Rapoport for generating approximate solutions of the free boundary problem for given obstacles [Ukrain. Mat. Žurnal 2, 107–117 (1950); these Rev. 12, 869]. The author extends the method to include obstacles with corners, and finds several conditions necessary for Rapoport's solution to be physically valid.

J. B. Serrin (Cambridge, Mass.).

Havelock, T. H. The moment on a submerged solid of revolution moving horizontally. Quart. J. Mech. Appl. Math. 5, 129–136 (1952).

Pond, Hartley L. The moment acting on a Rankine ovoid moving under a free surface. The David W. Taylor Model Basin, Washington, D. C., Rep. No. 795, x+17 pp. (1951).

Pond, Hartley L. The pitching moment acting on a body of revolution moving under a free surface. The David W. Taylor Model Basin, Washington, D. C., Rep. No. 819, viii+13 pp. (1952).

If a body of revolution moving in an infinite fluid with constant velocity in the direction of its axis can be represented by a source-sink distribution along the axis, the same distribution of sources and sinks beneath a free surface will give approximately the same body if the submergence is not too small. If the corresponding velocity potential is used to compute the forces and moments on the body, one may expect that the results will be somewhat in error. These three papers show that the error is important in the case of the moment about a transverse axis.

In the first paper above Havelock treats the case of a prolate spheroid moving with velocity c and computes a second approximation to the pitching moment. The contribution of the second term in the approximation is shown to equal $-(1+k_1)(1+k_2)^{-1}Rc^2/g$, where k_1 and k_2 are the inertia coefficients for longitudinal and transverse motion, respectively, and R is the (uncorrected) wave resistance. For large Froude number the contribution of this term dominates that of the first approximation, resulting in a negative (nose-down) moment for the body. For a more general body of revolution which may be represented by a source-sink distribution it is suggested that the formula above will serve as a suitable approximation for the corresponding term. For bodies of large length-diameter ratio this is further simplified to $-2Rc^2/g$ and the method applied to a Rankine ovoid, computations being shown graphically for two submergences. The factor $(-1)^s$ in the second displayed formula on p. 130 should be deleted according to the author.

In the second paper Pond computes the approximate moment on a Rankine ovoid, using essentially the same approximations as Havelock. Numerical calculations for two submergences are presented graphically. In the third paper he carries through the corresponding theory for a body of revolution, obtaining results equivalent to Havelock's. The work of the two authors is independent, however. The second order correction for the wave resistance is also included; an estimate shows it to be small compared to the first approximation. *J. V. Wehausen* (Providence, R. I.).

Ursell, F. Edge waves on a sloping beach. Proc. Roy. Soc. London. Ser. A. 214, 79–97 (1952).

After some preliminary discussion of the spectrum of a mechanical system, the author gives examples from the theory of gravity waves of three types of spectrum: (i) discrete (infinitely deep canal of finite length); (ii) continuous (infinitely deep canal of infinite length); (iii) mixed spectrum (semi-infinite canal with sloping bottom). It is pointed out that resonance can occur at an end-point of a continuous spectrum as well as at the values of a discrete one. The discussion of case (iii) generalizes the Stokes edge wave [cf. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, §260], exhibiting a discrete (finite, but increasing as the slope becomes small) spectrum of edge waves as well as a continuous spectrum. The effect of viscosity on resonance, especially for end-points of continuous spectra, is discussed. Some experimental confirmation is included. The author has pointed out that the last displayed formula on p. 95 should have the factor $\frac{1}{2}$ deleted, with corresponding changes in the following calculation. *J. V. Wehausen.*

Vallander, S. V. Computation of the flow about a cascade of profiles. Doklady Akad. Nauk SSSR (N.S.) 82, 345–348 (1952). (Russian)

The computation of a potential two-dimensional incompressible flow around a straight infinite cascade of profiles can be reduced to finding the (infinitely many-valued) analytic function $f(\zeta)$ mapping the covering surface of $|\zeta|>1$ with logarithmic branch-points at $z=\pm R$ $(R>1)$ onto the domain exterior to the profiles. The author proposes a new method for finding the function, analogous to a well-known formulation of the classical conformal mapping problem. In order to find $f(\zeta)$ it is sufficient to represent a profile parametrically in the form

$$x=P(R,\beta,\theta)+\sum A_n\cos n\theta-B_n\sin n\theta,$$
$$y=\sum B_n\cos n\theta+A_n\sin n\theta.$$

Here $0\leq\theta\leq 2\pi$; P is a given function, R, β, A_n, B_n, $n=1, 2, \cdots$, are unknown real constants. The author states, without details, that a modification of Nužin's method of conformal mapping [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 55–64 (1947); these Rev. 9, 113] yields a convergent procedure for finding these constants. *L. Bers.*

Polyahov, N. N. The flow about a grid of solid profiles of given form. Doklady Akad. Nauk SSSR (N.S.) **84**, 233–236 (1952). (Russian)

The problem treated here is the same as in the paper by Vallander reviewed above. While the present author's reasoning is different (and less transparent) he ends up with a similar system of equations. *L. Bers.*

Hopf, Eberhard. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. **4**, 213–231 (1951).

In his researches [(A) J. Math. Pures Appl. (9) **12**, 1–82 (1933); (B) ibid. **13**, 331–418 (1934); (C) Acta Math. **63**, 193–248 (1934)] on the integrals of the Navier-Stokes equations:

$$(1)\qquad \frac{\partial u_i}{\partial t}+u_\alpha\frac{\partial u_i}{\partial x_\alpha}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_\beta\partial x_\beta},\quad i=1,2,3,$$

(defining the motion of an incompressible viscous fluid), J. Leray introduced the following ideas: (a) the three components u_i of the velocity at every time $t\geqq 0$ belong to L^2 with respect to x (the kinetic energy is finite); (b) the equations (1) are replaced by the integro-differential equations of C. W. Oseen, which no longer contain the derivatives of second order with respect to x_i; (c) the derivatives of first order with respect to x_i could fail to exist, at least at certain times, and should be replaced by quasi-derivatives [whose definition is given in (C)]. In this paper, the author replaces (a) by the assumption that all functions involved belong to L^2 with respect to (x, t) and, by an elegant use of functions equal to zero outside of a bounded domain, replaces the integro-differential equations of Oseen by new ones in which the derivatives of first order of x_i have disappeared as well as the derivatives of second order. He considers an open domain $\hat{G}$ in the four-dimensional space (x, t) and the (real) Hilbert space H of all square-integrable functions in $\hat{G}$; he says that a vector field belongs to class N if it is equal to 0 outside of a compact set in $\hat{G}$; he then replaces equations (1) by

$$(2)\qquad \int\int_{\hat{G}}\left[\frac{\partial a_i}{\partial t}u_i+\frac{\partial a_i}{\partial x_\alpha}u_\alpha u_i+\mu\frac{\partial^2 a_i}{\partial x_\beta\partial x_\beta}u_i\right]dxdt=0$$

which must be satisfied for all vector fields $a_i\in N$ (and smooth enough). The class $H'(N)$ is defined by: $g(x, t)\in H'(N)$ if it is the limit of a weakly convergent sequence of functions $\gamma^{(n)}(x, t)$ belonging to N, having quasi-derivatives, the norm of the $\gamma^{(n)}$ and of their quasi-derivatives being uniformly bounded. Take the particular case in which $\hat{G}$ is a cylinder set: $x\in G$, $0<t<T$ where G is an arbitrary domain in the x space and define a class $H(N)$ in G in the same way as $H'(N)$ in $\hat{G}$. Obviously the fact that $g(x, t)\in H'(N)$ implies that at every time t in the interval $0<t<T$, $g(x, t)\to 0$ (in some generalized sense) when x converges to the boundary of G.

The main result of the present paper is the following existence theorem: Given a vector field $U_i(x)$ in G, satisfying the incompressibility condition and belonging to $H(N)$; there exists a vector field $u_i(x, t)$ in $\hat{G}$ with the following properties: (α) $u_i(x, t)\in H'(N)$; (β) $u_i(x, t)$ satisfies the equation (2); (γ) $u_i(x, t)$ converges strongly toward $U_i(x)$ if $t\to 0$. This theorem defines thus a "flow" of the fluid in G in a generalized sense (because the derivatives of the u_i do not need to exist) satisfying the given initial condition [property (γ)], the fluid being at rest on the boundary of G [property $H'(N)$]. *J. Kampé de Fériet* (Lille).

Bellman, Richard. On an equation occurring in the harmonic analysis of viscous fluid flow. Quart. Appl. Math. **9**, 218–223 (1951).

Consider the two-dimensional flow of an incompressible viscous fluid extending over the entire (x, y)-plane; under mild conditions of regularity it has been proved by the reviewer that the Fourier transform $z(w_1, w_2, t)$ of the vorticity $\zeta(x, y, t)$ satisfies a non-linear integro-differential equation [same Quart. **6**, 1–13 (1948); these Rev. **9**, 631]. Existence and uniqueness theorems seem difficult to establish for solutions of this equation corresponding to general initial conditions: $z(w_1, w_2, 0)=\varphi(w_1, w_2)$; the author considers the case in which the initial vorticity is small and proves: If $\max|\varphi(w_1, w_2)|$ is sufficiently small, there is a solution, which is unique and satisfies the inequality:

$$(1)\quad |z(w_1, w_2, t)|\leqq 8\max|\varphi(w_1, w_2)|/[1+v(w_1^2+w_2^2)t]^2$$

for all w_1, w_2 and $t\geqq 0$. By "sufficiently small" is meant that to each v there corresponds a constant $c(v)$ such that the condition $\max|\varphi(w_1, w_2)|\leqq c(v)$ suffices for the validity of (1). The proof of the existence of a solution satisfying (1) is based on successive approximations, proof of uniqueness is left to the reader. It is suggested that at the expense of decreasing $c(v)$ one can replace the exponent 2 by any arbitrary $n>2$ on the right side of (1). Obviously from (1) it follows that the solution $z=0$ is stable.

J. Kampé de Fériet (Lille).

***Targ, S. M. Osnovnye zadači teorii laminarnyh tečenil. [Fundamental problems of the theory of laminar flow.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 420 pp. 15.85 rubles.

This book is primarily a collection of various solutions or approximate solutions of the Navier-Stokes equations for special boundary conditions. Although it is not aimed at being encyclopedic in this respect, it covers most of the types of problems occurring in applications and includes several new solutions. Problems of special interest to mathematicians such as existence and uniqueness of solutions for more general boundary conditions are not discussed. Although a large part of the book is given over to approximate solutions, the pertinent exact solutions are usually given in each chapter, usually at the beginning. The chapter headings are as follows. 1) The equations of motion of an incompressible viscous fluid. 2) The simplest steady flows of a viscous fluid bounded by rigid walls. 3) Unsteady flows of a viscous fluid bounded by rigid walls. 4) Steady flow in a boundary layer. 5) Unsteady flow in a boundary layer. 6) Development of viscous flow in tubes. 7) Viscous flow in diffusors. 8) Flow of a viscous fluid in a lubricating film. 9) Some heat exchange problems for motion of a viscous liquid. *J. V. Wehausen* (Providence, R. I.).

Hannah, D. M. Forced flow against a rotating disc. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2772 (10,482)**, 17 pp. (1952).

This report treats the same problem as that of H. Schlichting and E. Truckenbrodt [Z. Angew. Math. Mech. **32**, 97–111 (1952); these Rev. **14**, 104]. Although it was written in 1947, it has just been published.

In contrast to the paper referred to above, this investigation makes use of the complete Navier-Stokes equations without boundary-layer approximations. Numerical methods were used to obtain solutions, which are tabulated and plotted against a dimensionless distance from the disc, for $\omega/a=0$, 1/2, 1, 2, ∞. The quantities given yield velocity

and pressure profiles, and torque. The displacement and momentum thicknesses are also evaluated, tabulated, and plotted. The author seems to have found agreement with Cochran for $a=0$, but she has corrected some of Homann's values for $\omega=0$. *W. R. Sears* (Ithaca, N. Y.).

Fogarty, Laurence Eugene. The laminar boundary layer on a rotating blade. J. Aeronaut. Sci. **18**, 247–252 (1951).

Boundary-layer equations for a thin blade of infinite span rotating steadily in an incompressible fluid are derived. At stations more than several chord lengths from the axis of rotation, the equations for the chordwise velocity are precisely the same as for the two-dimensional flow. Hence, in particular, the separation line is unaffected by rotation. When the chordwise velocity has been found, the spanwise flow can be calculated by solving a linear differential equation. Detailed numerical calculations are carried out for two examples: a flat plate and a symmetrical cylinder whose two-dimensional potential flow velocity is of the form $u_1=\text{const.}\times[(x/c)-(x/c)^3]$, c being the chord length. The effects of rotation are found to be small in both cases.

I. Imai (Tokyo).

Chandrasekhar, S. On turbulence caused by thermal instability. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 357–384 (1952).

The author points out that the principal feature of this type of phenomena is that a stationary axially symmetrical turbulence field can be maintained, and shows that, under stationary conditions, there is an exact balance between the dissipation of kinetic energy by viscosity and the liberation of potential energy by gravity. The author linearizes his equations at an earlier stage of the development of his theory.

The author's work on axially symmetrical turbulence is extended to the present case. It is shown that a Fourier analysis of the longitudinal correlation function in the direction of the gravitational field does not contain wavelengths less than a certain minimum value which is associated with the instability theory of Rayleigh and Jeffreys. Further, it is shown that the field of turbulence can be analyzed into two modes characterized by the kinetic energy being confined, principally to the vertical or to the horizontal direction. *C. C. Lin* (Cambridge, Mass.).

Bass, Jean. Les corrélations d'espace et de temps en turbulence homogène. C. R. Acad. Sci. Paris **232**, 1905–1906 (1951).

The author writes down the equations of correlation and spectrum for homogeneous turbulence in a uniform field of flow and discusses the use of Taylor's idea of connecting time and space correlations. *C. C. Lin.*

Corrsin, Stanley. The decay of isotropic temperature fluctuations in an isotropic turbulence. J. Aeronaut. Sci. **18**, 417–423 (1951).

The author develops the theory of isotropic temperature fluctuations in an isotropic field of turbulence analogous to the usual theory of velocity fluctuations. Analogous to Loitsiansky's relation, it is found that the second moment of temperature correlation remains invariant in time. For small Péclet numbers, the correlation equation reduces to that of the three-dimensional heat condition with spherical symmetry and the decay of mean-square temperature fluctuation follows the inverse 3/2 power of time. This relation was obtained earlier by E. Reissner in a direct analysis of the three-dimensional heat equation. It is stressed that the temperature spottiness always dies off at a slower rate than velocity spottiness. *C. C. Lin* (Cambridge, Mass.).

von Mises, R. Über einige Grundfragen der Hydrodynamik. Österreich. Ing.-Arch. **6**, 77–85 (1952).

The author studies the possible discontinuities in flows of a general class of ideal and real (viscous, thermally-conducting) fluids whose equations of motion include the usual Euler and Navier-Stokes equations as special cases. After defining what he means by a discontinuous solution of these equations, the author shows that the possible discontinuities in ideal fluids occur along contact and characteristic surfaces and that only the former type of discontinuity may occur in real fluids. Although the class of discontinuous solutions excludes shock waves, the author points out that any shock discontinuity can be obtained from the limit of appropriate solutions of the Navier-Stokes equations in which viscosity and thermal-conductivity approach zero simultaneously (with bounded ratio) [see Ludford, Quart. Appl. Math. **10**, 1–16 (1952); these Rev. **13**, 597]. He observes, however, that for a proper derivation of shock waves as the limit of flows of a real fluid, the latter flows should fulfill prescribed boundary conditions, unlike the solutions just mentioned. [This limit behavior has been proved by the reviewer [Amer. J. Math. **73**, 256–274 (1951); these Rev. **13**, 401] for steady one-dimensional flows of general fluids, in which viscosity and thermal conductivity approach zero in arbitrary manner.] *D. Gilbarg* (Bloomington, Ind.).

Skudrzyk, E. Die innere Reibung und die elastischen Eigenschaften fester, flüssiger und gasförmiger Körper. II. Österreich. Ing.-Arch. **6**, 157–196 (1952).

The author continus his studies of material defined by superposition of all orders of local time derivatives of stress and strain components [Acta Physica Austriaca **2**, 148–181 (1948); Österreich. Ing.-Arch. **3**, 356–373 (1949); these Rev. **11**, 283]. For infinitesimal oscillations of frequency ω, these equations are equivalent to those obtained by supposing the ordinary moduli of elasticity to be complex and frequency dependent [cf. Frenkel and Obrastzov, Acad. Sci. USSR. J. Phys. **2**, 131–142 (1940)]. The author discusses the phenomena represented in this theory. Among these are simple and complex relaxation effects in solids and fluids, elastic after-effects, viscous and plastic damping, hysteresis, and plastic flow. There are many comparisons with experimental data, which are used to determine experimental values for the various constants for various specific materials. A great deal of labor has gone into the work.

C. Truesdell (Bloomington, Ind.).

Byušgens, S. S. The geometry of adiabatic flow. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 30–51 (1951). (Russian)

This paper, dealing with stationary flows of an ideal weightless gas without friction and heat conduction, is a further exposition of the theory laid down in the previous articles [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 481–512 (1948); Doklady Akad. Nauk SSSR (N.S.) **58**, 365–368 (1947); these Rev. **10**, 633; **9**, 389]. The fundamental equations are the Euler equation $(\bar{v}\ \text{grad})\bar{v}+\rho^{-1}\ \text{grad}\ p=0$, continuity equation $\text{div}\ (\rho\bar{v})=0$, and the condition $p\rho_0^{-1}=\rho^k\rho_0^{-k}$. The method used is that of the mobile trihedron, built sometimes of oblique, sometimes of orthogonal vectors [see Buscheguennce, Izvestiya Akad. Nauk SSSR. Ser. Mat. **10**, 73–96 (1946); these Rev. **8**, 90]. The cases discussed include

those of minimal and rectilinear congruences, potential and spiral flows. A surface is called critical if $v^2-a^2=0$, minimal if div $(\bar{v}/v)=0$. Conditions are established for a congruence to be the congruence of streamlines of an adiabatic flow; these conditions involve the Mach number $\lambda=v^2a^{-2}-1$. When an adiabatic flow admits a minimal congruence of streamlines, it is called a minimal flow; only for such a flow is the scalar velocity constant along every individual streamline. When the streamlines are orthogonal to a family of isobaric surfaces ($\rho=$const.), then the flow is rectilinear.

D. J. Struik (Cambridge, Mass.).

Martin, John C. A vector study of linearized supersonic flow; applications to nonplanar problems. NACA Tech. Note no. **2641**, 81 pp. (1952).

Let $\nabla_\lambda=-\mathbf{i}\beta^2\partial/\partial x+\mathbf{j}\partial/\partial y+\mathbf{k}\partial/\partial z$ be the hyperbolic gradient operator, and let $\mathbf{n}_\lambda=-\mathbf{i}\beta^2\nu_1+\mathbf{j}\nu_2+\mathbf{k}\nu_3$ be the hyperbolic normal to a surface with ordinary normal $\mathbf{i}\nu_1+\mathbf{j}\nu_2+\mathbf{k}\nu_3$. For the equations (i) $\nabla\cdot\nabla_\lambda\varphi=0$ and $\nabla\cdot\nabla_\lambda\varphi=f$ the author has heuristically generalized various vector concepts used in the theory of Laplace's and Poisson's equations. For example, the divergence theorem takes the form

$$\text{(ii)}\qquad \int R^{-1}\nabla\cdot\nabla_\lambda\varphi\,dv=\oint(R^{-1}\nabla\varphi-\varphi\nabla R^{-1})\cdot\mathbf{n}_\lambda da,$$

where $R=[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2(z-\zeta)^2]^{\frac{1}{2}}$, and $\varphi=\varphi(\xi,\eta,\zeta)$. Applying (ii) for a solution of (i) to a region bounded by the Mach fore-cone of x, y, z and a surface S truncating this cone, and employing Hadamard's concept of the finite part of an integral, denoted by $^f\int$, yields

$$2\pi\varphi(x,y,z)=\,{}^f\!\int_S(R^{-1}\nabla\varphi-\varphi\nabla R^{-1})\cdot\mathbf{n}_\lambda da.$$

With this one can construct (anti-) symmetric solutions of (i) with prescribed values of $(\varphi)\partial\varphi/\partial z$ on $z=0$, where $(\partial\varphi/\partial z)\varphi$ is discontinuous. The author superposes such solutions with prescribed $\mathbf{n}\cdot\nabla\varphi$ on the planes of the fins to determine the damping in roll coefficient $C_{l_p}=\lim_{p\to0}\partial C_l/\partial(pb/2v)$ for rolling tails with 1, 2, 4, 6, or 8 rectangular fins of semi-span $\frac{1}{2}b$ in a supersonic stream of speed v, where C_l is the rolling moment coefficient per fin, and p is the rolling rate.

J. H. Giese (Havre de Grace, Md.).

Pai, S. I. On the flow behind an attached curved shock. J. Aeronaut. Sci. **19**, 734–742 (1952).

The differential equation for the stream function ψ of a plane or axially symmetric rotational flow of a perfect gas is linearized by the substitution

$$\psi=\psi_0(x,y)+\psi_1(x,y)+\psi_2(x,y)+\cdots,$$

where ψ_0 is a solution for irrotational motion, and ψ_1 is a first order perturbation of ψ_0. If the basic flow, given by ψ_0, can be taken uniform, then the equation for ψ_1 is particularly simple, and is appropriate for study of the flow behind a slightly curved shock. The author considers plane flow over an ogive in a supersonic stream. If the ogive is approximated by a wedge, then, as a first approximation, the shock may be assumed straight; the corresponding basic flow behind the shock is uniform. An iteration scheme is proposed for finding ψ_1 and the perturbed shock line. The first iteration is completed, and the results are analyzed.

J. B. Serrin (Cambridge, Mass.).

Meyer, R. E. On waves of finite amplitude in ducts. I. Wave fronts. Quart. J. Mech. Appl. Math. **5**, 257–269 (1952).

In part I, the behavior of waves travelling in steady isentropic flow in a duct of slowly varying cross-section is investigated. By integrating the equations of motion on the wave-front, the author shows that: (a) for a wave travelling upstream with respect to the moving fluid, if the original flow is supersonic and subsonic, respectively, in converging and diverging parts of the duct, with sonic speed at the throat, any disturbance at the exit or at the entry that starts with a sudden rise of local rate of change of pressure must lead to shock-formation after a finite time. If the flow is purely subsonic or supersonic in the duct, a compression wave may never develop into a shock provided the duct converges more rapidly than a cone for subsonic flow or diverges with cross-section area tending to infinity more strongly than the tenth power of the distance from the throat for supersonic flow. (b) For a wave travelling downstream with respect to moving fluid, a compression wave has a tendency to shock-formation for both types of steady flow mentioned above, but it will be blown out of the duct of finite length before the acceleration becomes large. If the duct extends indefinitely, the shock can still be avoided provided the duct diverges more rapidly than a cone for subsonic flow downstream the throat or the cross-section area tends to infinity more strongly than the tenth power of distance from the throat for supersonic flow downstream of the throat. Basically, the author's conclusions confirm those of Kantrowitz [NACA Tech. Note no. **1225** (1947); these Rev. **8**, 542].

Y. H. Kuo (Ithaca, N. Y.).

Meyer, R. E. On waves of finite amplitude in ducts. II. Waves of moderate amplitude. Quart. J. Mech. Appl. Math. **5**, 270–291 (1952).

The following problem is considered: If the steady flow in the duct is subsonic in the diverging part, a finite disturbance initiated during a time interval at the exit will propagate upstream into the steady flow. In the case of a weak wave (that is, the local velocity and speed of sound differ by first-order quantity from their steady values) a first-order solution is obtained. It is shown that to this order the solution does not distinguish between Mach lines of the disturbed flow and of the steady flow. Nevertheless, the limit line occurs and its position of occurrence in a time-space diagram is determined. From the first-order solution, a correction to the Riemann variables, namely a second approximation, is also obtained. By the knowledge of the first-order solution, the path of the shock can be constructed by a technique which is quite similar to that of Whitham [Proc. Roy. Soc. London. Ser. A. **201**, 89–109 (1950); these Rev. **12**, 298] in the case of steady supersonic flow over bodies of revolution.

Y. H. Kuo (Ithaca, N. Y.).

Newton, R. G. A progressing-wave approach to the theory of blast shock. J. Appl. Mech. **19**, 257–262 (1952).

In spherically symmetrical motion behind an expanding spherical shock wave the author assumes the velocity u, pressure p, and density ρ to take the forms $u=(r/t)W(\eta)$, $p=(r^{k+2}/t^2)\Pi(\eta)$, $\rho=r^k\Omega(\eta)$, where $\eta=r^{-\lambda}t$ and both λ and k are functions of the time t. These forms are substituted in the equations of motion. The shock wave (at which η is not assumed constant) is made to satisfy the Rankine-Hugoniot equations, and also the condition that the total internal and kinetic energy within it remains constant. Two simultaneous

differential equations of a very complicated appearance indeed for λ and the position of the shock wave as functions of time result. Solutions are attempted only as $t \to 0$ and as $t \to \infty$. The equations involve a number $\alpha = 2/3 - (k+5)/3\lambda$ which it is shown must be independent of time. Its value is inferred from considerations of the asymptotic behaviour of the wave as $t \to \infty$. At this stage it is assumed that in the limit as $t \to \infty$ the flow equations do not imply that any density or temperature gradients will be left. This assumption leads to the value $\alpha = -1$, and the conclusion that the shock strength falls off like $r^{-3/2}$. But there can be no doubt that the true asymptotic form is $r^{-1}(\log (r/r_1))^{-1}$, as given by all the standard theories [see, e.g., Brinkley and Kirkwood, Physical Rev. (2) **71**, 606–611; **72**, 1109–1113 (1947), whose method of approximate treatment of the problem remains the most plausible extant] and also by a quite different approach [Whitham, Proc. Roy. Soc. London. Ser. A. **203**, 571–581 (1950); these Rev. **13**, 180] derived directly from the equations of motion. The assumption mentioned is, of course, wrong. Entropy gradients must be set up as the shock decays, but the equations of motion permit only the equalization of pressure, taking no account of the diffusive processes which must ultimately (but negligibly slowly for the purposes of the shock wave propagation process) equalize also the temperature field. It appears that no simple modification will put the theory right asymptotically, and since the determination of α depends on the asymptotic behaviour, the whole solution must come under suspicion.

M. J. Lighthill (Manchester).

Shen, S. F. Note on the distribution of singular directions of a two-dimensional shock. J. Math. Physics **31**, 102–108 (1952).

For the flow over a pointed two-dimensional obstacle in a supersonic stream (Mach number $M > 1$) one may assume that an analytic body shape will lead to an analytic attached shock. Let α be the angle between the incoming stream and the shock. In a series of papers investigating this situation [e.g., same J. **28**, 62–90, 91–98, 153–172 (1949); these Rev. **10**, 758; **11**, 479], Thomas proved that one can expect stable analytic shocks only for $\alpha < \beta$, where $\beta = \beta(M)$ is the minimum angle giving subsonic flow behind the shock. Now let α_n, $n = 0, 1, \cdots$, be the smallest value of α for which $G_n(M, \alpha) = 0$ [see these Rev. **10**, 758]. The author shows that $\beta < \cdots < \alpha_n < \alpha_{n-1} < \cdots < \alpha_0 < \alpha_d$, where $\alpha_d = \alpha_d(M)$ is the shock angle giving maximum deflection to the incoming stream. The point of the paper is now the conjecture, made on the basis of the work of Guderley and Thomas, that $d^n k/ds^n$ exists only for $\alpha < \alpha_n$ (k = curvature, s = arc length, on the shock line). Thus, should one increase the vertex angle of the body past the point where subsonic flow beings, the derivatives of the shock line may run into trouble, proceding from the higher ones downward. This may be compared with the work of Cabannes [Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951); these Rev. **13**, 399].

J. B. Serrin.

Kawaguti, Mitutosi. Note on the velocity distribution over an elliptic cylinder submerged in a uniform flow of compressible fluid. J. Phys. Soc. Japan **7**, 313–315 (1952).

In an earlier paper [same J. **6**, 168–174 (1951); these Rev. **13**, 180] the author calculated to $O(M^4)$ the velocity distribution over an elliptic cylinder in symmetric flow in a stream of Mach number M. Here, to facilitate calculation of this distribution near the stagnation points, the analytic expressions of the earlier report are expanded in ascending powers of θ, the eccentric angle of the ellipse. Numerical values are given for the coefficients of θ and θ^2 for ellipses of various thickness ratios between 0 and 1.0. From the results it is concluded that the method of expansion in powers of M^2 converges rapidly near points of stagnation.

W. R. Sears (Ithaca, N. Y.).

Hasimoto, Hidenori. Application of the thin-wing-expansion method to the compressible flow past an elliptic cylinder. J. Phys. Soc. Japan **7**, 322–328 (1952).

This is a contribution to the study of the application of the method mentioned in the title to blunt-nosed cylinders, where questions of convergence arise because of the violation of the assumptions of the method. The method of expansion in powers of a thickness parameter was carried to the third approximation by Imai [Rep. Aero. Res. Inst. Tokyo Imp. Univ. No. **294** (1944)] by Imai and Oyama [Rep. Inst. Sci. Tech. Univ. Tokyo **2**, 37–39 (1948)] and by others. Here the Imai-Oyama theory is reviewed briefly and applied to elliptic cylinders in compressible stream flow at zero incidence. The velocity distribution over a cylinder of thickness ratio $t = 0.1$, at Mach numbers $M = 0.7$ and $M = 0.8$, is tabulated and plotted. In both cases, results are given to orders ϵ, ϵ^2, and ϵ^3, where ϵ is the thickness parameter $(2t/\mu)(1+\mu t)^{-1}$, $\mu = (1-M^2)^{1/2}$.

Also, making an approximation appropriate for thin ellipses, the formula for velocity distribution is rearranged formally as a power series in t, and the result, which includes terms of order t^3, is plotted and tabulated. The linear term in this series is the familiar Prandtl-Glauert approximation, while the quadratic term seems to agree with the results of Hantzsche and Wendt [Z. Angew. Math. Mech. **22**, 72–86 (1942); these Rev. **4**, 177], who calculated the flow about thin elliptical cylinders to the second order. For comparison, the velocity distributions for incompressible flow are plotted and also Imai's results (unpublished) for transonic flow and Kawaguti's [J. Phys. Soc. Japan **6**, 168–174 (1951); these Rev. **13**, 180; see also the preceding review] for the present Mach numbers, which are correct to $O(M^4)$. It is concluded that the present method unavoidably diverges near stagnation points, whether the expansion in t or the more accurate expansion in ϵ is used, but converges satisfactorily away from such points. The maximum velocity ratio, at the point of maximum thickness, is plotted against M, for four values of t, using the expansion in powers of t and showing contributions of orders t, t^2, and t^3 separately. All of these appear to be positive.

W. R. Sears.

Whitham, G. B. The flow pattern of a supersonic projectile. Comm. Pure Appl. Math. **5**, 301–348 (1952).

To obtain an approximation to the supersonic flow past a projectile of revolution which shall be equally good at all distances from the line of flight, the author introduces the hypothesis that in flow past a slender body linearized theory gives a correct approximation to the variation of the disturbances along a downstream characteristic provided that the departure of that characteristic from the straight is suitably taken account of. Strong arguments support this assumption, and it is checked successfully against known results at several places in the resulting theory. Except close to the body the flow pattern is determined by a function $F(y) = \int_0^y (y-t)^{-1} f'(t)\,dt$ where f is the axial source strength. It is shown how $F(y)$ can be obtained both for "smooth" slender bodies and for those with discontinuities in slope of their meridian section. Even for bodies which are not slender the theory holds for some $F(y)$, which cannot

then however be determined theoretically with good accuracy, but much useful information still results.

The characteristics are parabolas $x=\alpha r-kF(y)r^{\frac{1}{2}}+y$ (y=constant) with α, k functions of the main stream Mach number. On a given characteristic the disturbance velocities take the form $u=-F(y)(2\alpha r)^{-\frac{1}{2}}$, $v=-\alpha u$. The shock waves are determined from the approximate condition that they bisect the directions of the characteristics on each side of them. If at a given point the latter are $y=y_1$ and $y=y_2$, then the author deduces that the straight segment joining the points y_1 and y_2 on the graph of $F(y)$ must "cut off lobes of equal area on each side of the curve". The slope of the segment gives the distance r of the point on the shock from the axis; in fact the slope is $1/kr^{\frac{1}{2}}$. These two geometrical rules makes the determination of all the shocks in the flow an easy matter. It is carried out for a particular body-wake shape, a 5–10 calibre ogival head bullet with a wake turning through 12° at the base and straightening out to a radius half that of the body.

The shock shapes agree with the author's previous asymptotic theory [Proc. Roy. Soc. London. Ser. A. **201**, 89–109 (1950); these Rev. **12**, 298] at very large distances from the line of flight, and with the reviewer's result [Quart. J. Mech. Appl. Math. **1**, 309–318 (1948); these Rev. **10**, 413] at the conical tip of a slender body. The pressure-time curve recorded by an instrument at different distances from the line of flight is predicted. The part of the drag resulting from degradation of energy in the shock waves is determined as $\pi\rho U^2\int_0^\infty F^2(y)dy$ which is shown to agree with known expressions for the wave drag. *M. J. Lighthill.*

Van Dyke, Milton D. First- and second-order theory of supersonic flow past bodies of revolution. J. Aeronaut. Sci. **18**, 161–178, 216 (1951).

Various existing perturbation theories of supersonic flow past bodies of revolution are compared with known exact solutions, and it is concluded that the "first-order" theory without unnecessary approximations gives better results than commonly conceived. Using the cylindrical coordinates (x, r, θ) aligned with the body axis, the perturbation potential Φ satisfies the equation

$$\Box\Phi\equiv\Phi_{rr}+\Phi_r/r+\Phi_{\theta\theta}/r^2-(M^2-1)^{1/2}\Phi_{xx}=F[\Phi],$$

where $F[\Phi]$ is a cubic expression in Φ and its derivatives, and M is the free stream Mach number. The boundary condition at the surface of the body is

$$\Phi_r(x, R, \theta)+\sin\alpha\cos\theta=R'[\cos\alpha+\Phi_x(x, R, \theta)],$$

where $r=R(x)$ represents the meridian curve of the body and α is the angle of attack.

The "first-order" theory is concerned with the wave equation $\Box\varphi=0$ and the "second-order" theory with $\Box\varphi=F[\varphi]$. For axial flow the author solves completely the second-order problem, giving detailed computational procedures, but for inclined flow, the solution is not feasible except for a cone. The "slender-body" theory is based on the series expansion with respect to ϵ (and $\log\epsilon$), assuming $R(x)$ is of the order $O(\epsilon)$. Moreover, the following approximations are usually made: (i) approximate pressure relation $C_p=-2\Phi_x+\Phi_r^2$, (ii) approximate tangency condition $\Phi_r(x, R, \theta)+\sin\alpha\cos\theta=R'\cos\alpha$, and (iii) infinitesimal angle of attack $\sin\alpha\doteq\alpha$, $\cos\alpha\doteq 1$. According to the common view, the above simplifications are consistent with the first-order theory. But, comparing with the exact solutions for cones and characteristic solutions for some ogives, the author concludes that (ii) causes a negligible error and (iii) is unnecessarily restrictive, while (i) causes a drastic error and the slender-body theory reduces the accuracy considerably.

For inclined flow, $F[\Phi]$ contains terms linear in Φ. Surprisingly, however, the true "linearized" equation is shown to represent the flow less accurately than the first-order theory. A "hybrid" theory is then proposed, combining first-order cross flow and second-order axial flow. The result shows a definite improvement over first-order theory. Finally, non-linearity in lift is shown to result primarily from viscous separation of the cross flow. The resulting force can be estimated with reasonable accuracy with two-dimensional viscous sweepback theory. *I. Imai.*

Ferri, Antonio. The method of characteristics for the determination of supersonic flow over bodies of revolution at small angles of attack. NACA Rep. **1044**, 16 pp. (1951).

Supersedes NACA Tech. Note no. **1809** (1949); these Rev. **10**, 491.

Ferri, Antonio. Supersonic flow around circular cones at angles of attack. NACA Rep. **1045**, 11 pp. (1951).

Supersedes NACA Tech. Note no. **2236** (1950); these Rev. **12**, 875.

Nocilla, Silvio. Sul problema dell'ala triangolare a velocità ipersonica in deriva. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 159–171 (1951).

H. J. Stewart's work on the symmetrical linearized supersonic flow about a delta wing at a small angle of attack and with subsonic leading edges [Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109] has been extended to the yawed delta wing as follows. Let the undisturbed flow at Mach number M be parallel to the x-axis, and let the y-axis be coplanar with the wing. Let $\mu=[1-(M^2-1)(y^2+z^2)/x^2]^{\frac{1}{2}}$, $s=(1-\mu)^{\frac{1}{2}}(1+\mu)^{-\frac{1}{2}}$, $\theta=\arctan y/z$, and $\zeta=se^{i\theta}=\xi+i\eta$. Then the velocity components u, v, w are the real parts of analytic functions $U(\zeta)$, $V(\zeta)$, $W(\zeta)$. Here $W(\zeta)$ is determined by the conditions $w=0$ on $s=1$; poles of order one at the images ia and $-ib$ of the leading edges; and $w=w_0=$constant on $\xi=0$ for $-b<\eta<a$. Since $\zeta=(1+ip\zeta')/(ip-\zeta')$ with $(b-a)p=1-ab+(1-a^2)^{\frac{1}{2}}(1-b^2)^{\frac{1}{2}}$ maps $|\zeta|\leqq 1$ onto $|\zeta'|\leqq 1$ and transforms ia and $-ib$ into symmetrical points $\pm is_0$, the determination of $W(\zeta(\zeta'))$ in the ζ'-plane has been reduced to a problem solved by Stewart. By the conical flow relation $dU/d\zeta'=-2\zeta(1+\zeta^2)^{-1}dW/d\zeta'$, $U(\zeta')$ can be expressed as a linear combination of Stewart's function $U(\zeta')$ and another elementary function of ζ'. The author has derived formulae for the lift and moment coefficient slopes $dC_L/d\alpha$ and $dC_M/d\alpha$, and has plotted numerical results for two wings at $M=\sqrt{5}$ as functions of yaw angle.

J. H. Giese (Havre de Grace, Md.).

Moreau, Jean-Jacques. Sur la théorie tourbillonnaire du profil portant en régime non stationnaire. C. R. Acad. Sci. Paris **235**, 695–697 (1952).

Shen, Shan-fu. An estimate of viscosity effect on the hypersonic flow over an insulated wedge. J. Math. Physics **31**, 192–205 (1952).

For hypersonic flows over a wedge, the shock angle will generally be small. If the shock near the leading edge is assumed to coincide with the edge of the boundary layer, there will be expected a region of interaction between the viscous effect and the shock. Assuming that the boundary-layer equations are valid and that the velocity distribution inside

the boundary layer is linear for the case of insulated wall, the momentum integral can be integrated. It is shown that in this case the boundary layer grows asymptotically as $Kx^{3/4}$, K and x being respectively the similarity parameter and the distance from the leading edge, and the pressure varies asymptotically as $K^3x^{-1/2}$. An interaction distance is defined and is calculated as a function of the similarity parameter. A similar investigation for a flat plate is also carried out. *Y. H. Kuo* (Ithaca, N. Y.).

Tsien, H. S. Influence of flame front on the flow field. J. Appl. Mech. **18**, 188–194 (1951).

The first part of this paper is devoted to an analysis of a curved flame front analogous to the usual theory of a curved shock. The flame front is taken to be either constant or proportional to the absolute temperature. It is shown that considerable vorticity is produced by combustion.

The author then considers the problem of the flame width in a two-dimensional combustion chamber of constant width with a flame holder at the center. A very simple method is given to reproduce the results obtained by Scurlock by elaborate numerical calculations. This method is then extended to take into account the compressibility of the gas. While the width distribution as a function of the completeness of burning is found to be little influenced by compressibility, anomalous behaviour is indicated by the occurrence of maximum width at partial burning. The author discusses the physical situation related to this anomalous behaviour and its relation to the use of combustion chambers of variable widths at high Mach numbers.
C. C. Lin (Cambridge, Mass.).

Stocker, P. M. The transients arising from the addition of heat to a gas flow. Proc. Cambridge Philos. Soc. **48**, 482–498 (1952).

The development of the one-dimensional flow pattern for the case of a uniform rate of entropy addition over a limited region may be solved with some difficulty by the method of characteristics. The authors propose two linearizing approximations to investigate the transients preceding the steady state discussed by Foa and Rudinger [J. Aeronaut. Sci. **16**, 84–94, 119 (1949)]. The first assumes linear characteristics of constant slope which are sufficient to map out the transient flow regions and the pattern of transition to steady state. This, however, introduces serious errors at moderate Mach numbers in the behavior of the receding characteristics. An improved approximation leaves the advancing characteristics linear but introduces a first order correction to the receding characteristics using the rate of entropy addition as an approximation parameter. The several transient regions formulated explicitly for the linear approximation may still be described in explicit form with several examples indicated. This first order correction is shown to correct most of the error inherent in the linear approximation. *N. A. Hall* (Minneapolis, Minn.).

van der Does de Bye, J. A. W., and Schenk, J. Heat transfer in laminary flow between parallel plates. Appl. Sci. Research A. **3**, 308–316 (1952).

In determining the temperature distribution and resulting heat flow in two-dimensional laminar flow, the thermal resistance of the wall is significant. Defining a wall Nüsselt number as $\mathrm{Nu}'=UD/k$ where U is the overall coefficient of thermal conductivity for the wall, previous work is extended from $\mathrm{Nu}'=0$ and ∞ to the specific cases of $\mathrm{Nu}'=2$ and 20. This is sufficient to indicate the general behavior. The solution requires determination of eigenvalues for the equation $X''+(1-4x^2)X=0$ with the boundary condition $\mathrm{Nu}'\cdot X+X'=0$ at $x=\frac{1}{2}$. These are tabulated. The nature of the extension to axial symmetric flow is indicated.
N. A. Hall (Minneapolis, Minn.).

Monin, A. S. Variation of pressure in a barotropic atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1952**, no. 4, 76–85 (1952). (Russian)

The atmospheric model determined by the equations of frictionless inviscid motion, the hydrostatic equation, the equation of continuity, and the polytropic equation of state is investigated, using the method of dimensionless parameters. The vertical component of motion is shown to be directly proportional to the horizontal divergence and to the height; the direction of vertical motion is determined by the horizontal divergence and the vertical lapse rate of temperature, upward for convergence in an adiabatic atmosphere. Local pressure tendency can be expressed in terms of the local variation of vertical motion with height. Potential vorticity is shown to be conservative. Finally, the method of Kibel' [Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **1940**, no. 5] is applied to obtain a first approximation to the pressure tendency in the base pressure field in terms of the field itself and its derivatives.
W. D. Duthie (Monterey, Calif.).

Stümke, H. Zur Berechnung der Drucktendenz bei Wärmezufuhr innerhalb einer isothermen Atmosphäre von konstanter Grundgeschwindigkeit. Z. Angew. Math. Mech. **32**, 68–75 (1952).

L'auteur propose une méthode de calcul de l'évolution des atmosphériques de proche en proche, en introduisant dans les équations aérodynamiques, les variations moyennes de la pression, de la densité, de la température, et de la vitesse. Ceci conduit à une équation différentielle elliptique que doit vérifier le régime local de la pression, en tenant compte de trois paramètres fondamentaux (latitude, température, vitesse). En supposant le terrain plat, cette équation se résout dans le demi-espace à l'aide d'une fonction de Green. Ce cas diffère de celui de l'atmosphère calme au début, étudié par Prandl, par la présence d'un terme représentant la variation d'énergie cinétique, ce qui est important du point de vue météorologique. Cette solution n'est valable que localement du point de vue temporel, et pour continuer il faut reprendre les calculs après un laps de temps suffisamment petit. *V. Z. Kostitzin* (Paris).

Guptill, E. W., and MacDonald, A. D. The acoustical field near a circular transducer. Canadian J. Physics **30**, 119–122 (1952).

Gli autori fanno uso dell'espressione

$$\Phi=\int_0^\infty \frac{g(\alpha)J_0(\alpha\rho)e^{-i\mu z}}{i\mu}\alpha d\alpha \quad (\mu=(k^2-\alpha^2)^{1/2})$$

del potenziale di velocità, cui corrisponde per $g(\alpha)$ l'espressione

$$g(\alpha)=\int_0^\infty V(\rho)J_0(\alpha\rho)\rho d\rho$$

essendo V la velocità delle particelle per $z=0$. Viene data una soluzione approssimata per il campo in vicinanza di un diaframma circolare fissato lungo il suo perimetro e sollecitato da una pressione oscillante eguale in tutti i suoi punti. Quando tale trasduttore è usato come ricevitore, la velocità in prossimità di esso risulta maggiore che per l'onda piana.
G. Toraldo di Francia (Firenze).

Berry, F. J. The diffraction of sound pulses by an oscillating, infinitely long strip. Quart. J. Mech. Appl. Math. **5**, 324–332 (1952).

The author obtains the solution of the problem of the diffraction of a plane pressure pulse normally incident on an infinitely long strip of finite width, capable of motion as a spring-supported rigid body. The solution is based on the solution of the diffraction of an incident pulse of general form by a stationary strip, given by E. N. Fox [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 71–103 (1948); these Rev. **10**, 166]. Numerical calculations have been made for the case when the incident pulse is sharp-fronted and of constant unit pressure; and graphs show how the mobility of the strip affects the pressure distribution on the back of it.

E. T. Copson (St. Andrews).

Berry, F. J. The diffraction of a sound pulse by a non-rigid semi-infinite plane screen. Quart. J. Mech. Appl. Math. **5**, 333–343 (1952).

The problem considered in this paper is the two-dimensional problem in which a plane-fronted sound pulse is incident normally on a semi-infinite plane screen. Usually the screen is assumed to be rigid and stationary, which is by no means an accurate representation of what happens in practice. Here the screen is assumed to be non-rigid in that each particle of it is capable of yielding independently under the action of the pulse. [Cf. the preceding paper where diffraction by a spring-supported rigid body is considered.] An integral equation is formed for the pressure saltus across the screen using a method due to F. G. Friedlander [same J. **4**, 344–357 (1951); these Rev. **13**, 407]. Two methods of solving the integral equation are given, both depending on successive approximations. The results found by the first method are illustrated graphically. *E. T. Copson.*

Elasticity, Plasticity

***Legendre, Robert. Fonctions biharmoniques doublement complexes et applications aux problèmes d'élastostatique.** Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. **42**, iv+45 pp. (1951).

The author introduces the concept of a doubly-complex function $L=Z(z,t)+jT(z,t)$, where $j^2=0$, and $\partial Z/\partial t=0$, $\partial Z/\partial z=\partial T/\partial t$, and develops some of its analytical properties. It is shown that to each biharmonic function there is a corresponding doubly-complex function. In this way, starting with the biharmonic Airy stress function of a plane stress problem, one is led to formulate the solution of the problem in terms of the corresponding doubly-complex "stress function", with the aid of which the stresses and displacements may be expressed. The last two chapters are devoted to the study of the plane problems of a hole in an infinite plate and of a half-plane subjected to forces along part of its boundary. *J. B. Diaz* (College Park, Md.).

Stephens, Kathleen M. A boundary problem in orthotropic generalized plane stress. Quart. J. Mech. Appl. Math. **5**, 206–220 (1952).

The author employs the function-theoretic method of Livens and Morris [Philos. Mag. (7) **38**, 153–179 (1947); these Rev. **9**, 121] to study a generalized plane stress problem in an orthotropic plate which contains a curvilinear triangular hole. The region external to the hole is mapped onto the region exterior to a unit circle in a new plane. The actual determination of the complex stress function requires a modification of Morris' method. This function is obtained to a suitable degree of approximation. It is applied to an orthotropic plate subjected to constant directed tension and shear, also to a radial tension on the infinite boundary. For these cases the hoop stresses on the boundary of the hole are given for spruce and oak plates. In another case where constant normal and tangential loads are applied on the boundary of the hole, the hoop stresses are given for the isotropic plate and at significant points for oak and spruce plates. *D. L. Holl* (Ames, Iowa).

Tiffen, R. Boundary-value problems of the elastic half-plane. Quart. J. Mech. Appl. Math. **5**, 344–351 (1952).

In this investigation we employ a combination of complex potential and Fourier integral methods to solve problems of generalised plane stress of semi-infinite plates with one straight boundary. In the first part of this paper relations between the complex potentials $\Omega(z)$, $\omega(z)$ are obtained suitable for giving any desired conditions of stress or displacement along the real axis, thus reducing problems of the half-plane $y\geqq 0$ to the determination of the single potential function $\Omega(z)$. A simple method is given of obtaining $\Omega(z)$ from specified conditions along the real axis and at infinity. The method is applied to the solution of problems of the half-plane $y\geqq 0$ in equilibrium with specified stresses or displacements along the straight boundary and balancing stresses at infinity and to the problem of the half-plane subject to an interior force and with a straight boundary free from displacement. (From the author's summary.)

R. M. Morris (Cardiff).

Tiffen, R. Solution of two-dimensional elastic problems by conformal mapping on to a half-plane. Quart. J. Mech. Appl. Math. **5**, 352–360 (1952).

In this paper we give a general method of solution of problems of generalised plane stress of plate-like material bounded by a single unclosed contour. The theory is restricted to material of such a form that conformal mapping onto a half-plane is possible. That part of the boundary which is mapped onto the straight boundary of the half-plane is subject to specified stresses or displacements, equilibrium being maintained by evanescent stresses at infinity. The method is illustrated by consideration of material with a parabolic boundary subject to simple boundary loading. (From the author's summary.) *R. M. Morris.*

Položiĭ, G. N. The solution of some problems of the plane theory of elasticity for regions with angular points. Ukrain. Mat. Žurnal **1**, no. 4, 16–41 (1949). (Russian)

Using the author's terminology, in the third boundary value problem of the plane theory of elasticity the normal displacement and the tangential stress are prescribed on the boundary, while in the fourth boundary value problem the tangential displacement and the normal stress are prescribed on the boundary. The third boundary value problem, when the tangential stress is zero, was considered by Mushelišvili [Certain fundamental problems in the theory of elasticity, Leningrad, 1933, p. 216; Moscow, 1935, pp. 303–318; Doklady Akad. Nauk SSSR (N.S.) **3**, 7–11 (1934)] and solved by him for domains which are mapped on a circle by means of rational functions. For domains with smooth boundaries this problem was solved by D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. **7**, 413–420 (1943); these Rev. **6**, 195] but it has not yet been discussed in general for

domains with boundaries having angular points. In the present paper the author solves the third and fourth boundary value problems for a rectangle, acute-angled triangle and right triangle, under certain hypotheses concerning the growth of the stresses near the angular points.

J. B. Diaz (College Park, Md.).

Položiĭ, G. N. Application of boundary problems of the theory of functions to the solution of the third problem of the plane theory of elasticity for an infinite plane with triangular and regular polygonal openings. Ukrain. Mat. Žurnal **2**, no. 3, 115–124 (1950). (Russian)

The third boundary value problem of the plane theory of elasticity consists in the determination of the stresses and displacements in the interior of a plane domain when the normal component of the displacement and the tangential stress are prescribed on the boundary of the domain. The solution of this problem, for certain domains bounded by piecewise-rectilinear curves was given earlier by the author [Doklady Akad. Nauk SSSR (N.S.) **66**, 353–356 (1949); Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 297–306 (1949); these Rev. **11**, 68, 285]. The present paper contains the solution of the third boundary value problem for the domains mentioned in the title. *J. B. Diaz.*

Yu, Yi-Yuan. Heavy disk supported by concentrated forces. Quart. Appl. Math. **10**, 280–284 (1952).

Following the method of Mushelišvili [Z. Angew. Math. Mech. **13**, 264–282 (1933), pp. 273–274] for the problem of a two-dimensional light circular disc subject to an arbitrary number of concentrated forces on the boundary, the author extends the results to the case of a heavy disc, by the inclusion of body forces. As special cases treated by other writers he gives the results for (i) a heavy disc resting on a horizontal plane, (ii) a heavy disc resting on the ends of its horizontal diameter. *R. M. Morris* (Cardiff).

Kalandiya, A. I. On a mixed problem of the bending of an elastic plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 271–282 (1952). (Russian)

The boundary value problem considered is the following: to determine the deflection $u(x, y)$ of a thin elastic plate occupying a plane open set T, when the plate is clamped along a part of the boundary of T, and simply supported along the remainder of the boundary of T. Analytically, $u(x, y)$ satisfies $\Delta\Delta u = q/D$ in T; $u = \partial u/\partial n = 0$ on part of the boundary of T, and

$$\sigma\Delta u + (1-\sigma)[u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta] = 0,$$

on the remainder of the boundary of T; where q is a given function, D and σ are elastic constants, Δ is the Laplacian, n is the outer normal, and θ is the angle between the outer normal and the x-axis. By employing an integral representation due to D. I. Šerman [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 28–31 (1940); these Rev. **2**, 270] the author reduces the boundary value problem in question to an equivalent system of Fredholm integral equations, and discusses the existence and uniqueness of the solution of these integral equations. *J. B. Diaz.*

Berger, E. R. Zur Berechnung des Einflussfeldes der quadratischen Platte. Ing.-Arch. **20**, 207–210 (1952).

Pucher [Ing.-Arch. **12**, 76–100 (1941); these Rev. **10**, 86] has treated the case of the influence field for the supporting couple at the centre of one side of a square clamped plate. He pointed out that the expression

$$(1) \qquad w = \frac{1}{\pi}\left(-\cos^2\varphi + \sum_{n=1}^{\infty} c_n \rho^{n+1}\cos\varphi\cos n\varphi\right)$$

for the calculation of this influence field fails, in spite of the fact that the series is a solution of the biharmonic equation $\Delta\Delta w = 0$. He went on to use, instead of the above trigonometric series, a series of biharmonic polynomials. In this paper Berger shows that the series in (1) fails because it is incomplete and goes on to show that the expression

$$w = \frac{1}{\pi}\left[-\cos^2\varphi + \sum_{n=1,3,5,\cdots} c_{n+1}\rho^{n+1}\cos\varphi\cos n\varphi + \sum_{n=2,4,6,\cdots} c_{n+1}\rho^{n+1}\left(\cos\varphi\cos n\varphi + \frac{1}{n}\sin\varphi\sin n\varphi\right)\right]$$

is the one required, and he is able to identify this series with the biharmonic polynomials of Pucher. As pointed out by Pucher in a postscript to the paper, other methods have now been found which are more satisfactory for calculating the influence field numerically, but Berger's result is of theoretical interest from the point of view of the solutions of the biharmonic equation. *R. M. Morris* (Cardiff).

Gol'denveĭzer, A. L. On the application of the solutions of the Riemann-Hilbert problem to the calculation of momentless shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 149–166 (1951). (Russian)

The displacements and tangential stresses of a momentless shell whose middle surface is in the form of an ellipsoid, two sheeted hyperboloid, or elliptic paraboloid are expressed in terms of the real and imaginary parts of complex-valued analytic displacement and stress functions, respectively, whose determination leads to the solution of Riemann-Hilbert problems of the theory of analytic functions [N. I. Mushelišvili, Singular integral equations . . . , Moscow-Leningrad, 1946; these Rev. **8**, 586]. *J. B. Diaz.*

Oniašvili, O. D. On the computation of sloping shells with a horizontal force. Soobščeniya Akad. Nauk Gruzin. SSR. **12**, 103–110 (1951). (Russian)

A shell supported radially at the edges presents the simplest boundary problem and it is solved by a well-known double trigonometric series. If only one of the edges is not supported radially the case is not so simple. In such a case the author recommends using a single trigonometric series or a variational method. The use of a single trigonometric series is illustrated on three examples. In the first example we have a spherical momentless shell with two radially supported edges, the supports of the remaining two edges are arbitrary. The load distributed on the entire surface is continuous, constant, and directed radially. In the second example the same spherical shell with the same supports has horizontal continuously distributed load. The third example deals with a cylindrical shell loaded by horizontal forces. The bending moments in this case are not neglected. The supports are as in the above two examples. The use of a variational method is illustrated on the last example. The author solves a case of a shell whose shape and supports are arbitrary. He utilizes in this problem the functions which are often used for transverse vibrations of a beam.

T. Leser (Lexington, Ky.).

Oniašvili, O. D. On vibrations of a sloping cylindrical shell. Soobščeniya Akad. Nauk Gruzin. SSR. 9, 425–431 (1948). (Russian)

The shell under consideration has the shape of a cylindrical arch; it is cut out from a cylindrical shell with diaphragms at both ends and is supported on hinges all around the edges. The system of two differential equations and the expression for the frequency of free vibrations of a cylindrical shell with an infinite number of degrees of freedom derived from the above equations were given by V. Z. Vlasov [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 27–52; these Rev. **10**, 653].

The author of this paper uses Vlasov's equations and finds the principal mode of free vibrations for a shell (a) initially stressed, (b) free of stress, and (c) the modes of vibrations forced by a periodic load distributed continuously on the surface of the shell. The last case is discussed in detail for the following subcases: (i) the force is directed radially and the magnitude is a simple periodic function of time, independent of the location on the surface of the shell; (ii) the external load is a concentrated periodic force directed radially. The vibrations due to external loads are considered separately for the case when the free vibrations are damped out completely, and for the case when the free vibrations are present and the resulting oscillations are not periodic, beating being the most important occurrence. In this latter case the author obtains the period of the beats.

T. Leser (Lexington, Ky.).

Weber, C. Kugel mit normalgerichteten Einzelkräften. Z. Angew. Math. Mech. **32**, 186–195 (1952). (German. English, French and Russian summaries)

This paper treats, within the classical theory of elasticity, the problem of an elastic sphere under radial concentrated surface loads, and deals in particular with the case of two equal and opposite loads applied to the endpoints of a diameter of the sphere. The same problem was also treated concurrently and by different means, in a paper by F. Rosenthal and the reviewer [J. Appl. Mech. **19**, 413–421 (1952)]. This latter paper will be referred to as (B) in what follows. A three-dimensional photoelastic investigation of the problem was carried out by M. M. Frocht and R. Guernsey, Jr.

The series solution obtained by the author rests on an adaptation to the loading conditions under consideration of the general solution to the problem of the sphere given by E. Trefftz [Handbuch der Physik, vol. VI, Springer, Berlin, 1928, p. 102]. Trefftz's approach, which, in turn, utilizes a scheme apparently due to E. Almansi [Mem. Accad. Sci. Torino (2) **47**, 103–125 (1897)], in the case of an axisymmetric loading, reduces the problem to the solution of two Dirichlet problems for the sphere. In the special instance of two equal and opposite concentrated loads, acting at the poles of the sphere, the approach necessitates the determination of but a single load function Z which is harmonic inside the sphere and vanishes on the boundary except at the poles where it must possess suitable singularities. The particular specification of these singularities, given in the paper, is not clear to the reviewer. In determining Z the author first writes an analogous load function for the half-space under a concentrated normal load, and then applies an inversion followed by a symmetrization about the equator plane of the sphere; no justification is given for this ingenious procedure. The correctness of the final formula for Z, however, may readily be confirmed by an appropriate limit process applied to the case of tractions uniformly distributed over two spherical caps, each surrounding one of the poles. Indeed, it is such a limit process which provides an ultimate criterion for the validity of solutions to concentrated-force problems and precludes the possibility of pseudo-solutions, the existence of which is demonstrated in (B).

The paper unfortunately abounds with misprints and contains two errors. The leading term in the denominator of Eq. (11) should read $(n-1)(n-2)$ instead of $(n-1)^2$. This error is carried through the remaining computations, and a second independent error is introduced in Eq. (23): the first factor in the numerator of the second fraction should be $(2n+5)$ instead of $(2n+3)$. As a consequence, many of the subsequent formulas, and in particular the closed expressions for the principal stresses at the center of the sphere, given at the end of the paper, are incorrect.

It should be emphasized that the problem under consideration may be approached through any one of the various available solutions to the general problem of the sphere. The singularities encountered in the presence of concentrated loads, however, require special attention if the solutions corresponding to such singular loading conditions are to be practically useful. Thus the series for the stresses obtained in the present paper are divergent at the surface of the sphere. With a view toward remedying this deficiency the author removes by subtraction in closed form, at each pole of the sphere, the Boussinesq singularity appropriate to a concentrated load acting perpendicular to a plane boundary. This process, which is illustrated only for the first invariant of stress, will be found to leave the series for the individual stress components still divergent at the surface; numerical evaluations of the stresses near the boundary remain, therefore, virtually impossible. The reason for this difficulty lies in the fact that the singularities inherent in the current problem are not identical with the Boussinesq singularity, as is shown in (B). A more thorough treatment of the problem requires the determination in closed form of the proper singularities at least to the extent where the residual problem is characterized by finite and continuous surface tractions.

E. Sternberg (Chicago, Ill.).

Ručimskiĭ, M. N. On the computation of a beam acted on by a moving force of variable mass. Akad. Nauk SSSR. Inženernyĭ Sbornik **11**, 187–188 (1952). (Russian)

Tyler, C. M., Jr., and Christiano, J. G. An Airy integral analysis of beam columns with distributed axial loading having a fixed line of action. J. Appl. Mech. **19**, 275–283 (1952).

An analytic method of solution of beam columns with partial uniformly distributed axial loading, whose line of action does not change as the column deflects, is developed in terms of Airy integral functions. The method is valid when the beam bending moment can be expressed or approximated piecewise as a polynomial. The case of a beam column having a part tension axial loading, supported at its ends, and having a transverse beam load, i.e., a discontinuous transverse loading, is treated in detail, the tables of the necessary integral functions and their first derivatives being included. The method can also be applied to non-uniformly distributed axial loading.

R. M. Morris (Cardiff).

Reissner, Eric. On non-uniform torsion of cylindrical rods. J. Math. Physics **31**, 214–221 (1952).

The author applies a variational theorem [Reissner, same J. **29**, 90–95 (1950); these Rev. **12**, 301] to non-uniform tor-

sion of cylindrical rods where restraint of warping of the end section is assumed. The theorem admits of applying independent variations to both stresses and displacements which is in contrast to other minimal principles involving variations of either stresses or displacements but not both. Following St. Venant theory, τ_{xx}, τ_{yy}, and τ_{xy} are to vanish everywhere, and the displacements are $u=\beta(z)y$, $v=-\beta(z)x$. The added restraint of no warping at the fixed end $z=0$ is considered under three types: (A) Set $w=\beta'(z)\phi(x, y)$ where ϕ is not necessarily the St. Venant warping function; (B) set $w=\alpha(z)\phi(x, y)$, where $\alpha(z)$ is not identical with $\beta'(z)$; (C) set $\tau_{zz}=E\partial w/\partial z=\sigma(z)\phi(x, y)$. In (A) the variational method produces $\beta'(0)=0$, $\beta''(L)=0$, $\tau_{zz}=E\phi\beta''$ and a differential equation for $\beta(z)$. In (B), $\alpha(0)=0$, $\alpha'(L)=0$, $\tau_{zz}=E\phi\alpha'$, $\beta'(0)=\bar{T}/GI_p$ ($\bar{T}$=applied torque, G=shear modulus, I_p=polar inertia), and there are two simultaneous equations for $\alpha(z)$ and $\beta(z)$, with $\beta''(L)=0$, where L is length of prism. In (C) two differential equations determine $\sigma(z)$ and $\beta(z)$ with appropriate end conditions at $z=0$. Some valuable comparisons are made for the three cases. *D. L. Holl.*

Trösch, A. Stabilitätsprobleme bei tordierten Stäben und Wellen. Ing.-Arch. **20**, 258–277 (1952).

The author treats the buckling of rods and critical speeds of rotating shafts under the action of external twisting moments and direct axial loads. These external loads are to maintain a constant axial direction in contrast to the study of Ziegler [Z. Angew. Math. Physik **3**, 96–119 (1952); these Rev. **13**, 797] where a conservative system is present. The buckling problem leads to various forms of characteristic equations depending upon the end conditions of the rod, the single or combined action of thrust and torsion, and upon the presence of a damping factor. In the rotating shaft, it is assumed that the mass of the shaft is negligible in comparison with the massive pulley on the shaft. Some cases of end support make the shaft highly unstable; however, added damping or the inclusion of gyroscopic terms in the analysis greatly alters the early instability. *D. L. Holl.*

Bellin, E. A., and Džanelidze, G. Yu. Survey of papers on the dynamical stability of elastic systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 635–648 (1952). (Russian)

***Brodeau, André. Vibrations des solides déformables, isotropes et anisotropes, homogènes et hétérogènes.** Publ. Sci. Tech. Ministère de l'Air, no. **254**, Paris, 1951. vi+146 pp.

This book is almost exclusively confined to plane waves in three-dimensional infinite elastic media, and to waves in plates and rods analysed approximately by a similar type of analysis. The homogeneous isotropic case is considered first, and the wave equations for pure dilatational and pure rotational waves are deduced. Thin plates are considered, assuming motion only in the plane of the plate, and tubes and rods by allowing approximately for the radial motion. The exact theory is given for longitudinal sinusoidal waves in a solid circular cylinder. Slabs of different materials with plane parallel contact surfaces are analysed for waves propagated normally to these surfaces. Non-homogeneous material is considered, and an approximate analysis presented for variation of material properties in one direction which coincides with that of wave propagation.

Fibrous bodies with a fixed fibre direction determining anisotropy, cubical anisotropy, more general combinations of these, and general anisotropy are considered with the orientation of material properties constant throughout the body. Plane waves are considered, and the propagation directions in which these are possible are investigated. Some experimental data are presented for rubber with varying amounts of filler.

Equations for all stress and displacement components are printed in full throughout the book so that long series of equations appear which could have been avoided by an appropriate notation or non-repetition of equations merely involving cyclic repetition. *E. H. Lee.*

Kupradze, V. D. Solution of a fundamental boundary problem in the displacements for vibrations of an elastic medium. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 99–106 (1948). (Russian)

The boundary value problem consists in the determination of the displacement vector $\bar{u}(u_1, u_2, u_3)$ which takes prescribed boundary values $\bar{f}(f_1, f_2, f_3)$ on the boundary S of an open set B in (x_1, x_2, x_3)-space, satisfies the system of equations

$$(*) \qquad \Delta\bar{u}+\frac{\lambda+\mu}{\mu}\operatorname{grad}\operatorname{div} u+k^2\bar{u}=0,$$

in B, and the radiation condition at infinity if B is unbounded. The uniqueness theorem in the plane case has been discussed earlier [Kupradze, C. R. (Doklady) Acad. Sci. URSS (N.S.) **6** (1935 II), 100–104; and in the space case in a Tiflis dissertation by A. S. Bakalyaev]. In the present paper the author is concerned with the existence of a solution. A "fundamental vibration tensor solution" of (*) is introduced, and "double layer potentials" with respect to this tensor are then constructed. Using the jump conditions on the boundary for such double layer potentials, the solution of the (Dirichlet-type) boundary-value problem is sought in the form of a double layer potential, and an equivalent Fredholm integral equation for the unknown density of the double-layer potential is obtained. For the exterior boundary-value problem the solution exists for any values of the vibration parameter. *J. B. Diaz.*

Vekua, I. N. On the proof of some uniqueness theorems occurring in the theory of steady vibrations. Doklady Akad. Nauk SSSR (N.S.) **80**, 341–343 (1951). (Russian)

V. D. Kupradze [see, e.g., Boundary problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950], by using a certain positive definite form, called the quasi-energy, has shown the uniqueness of the solution of certain boundary value problems in electromagnetic and elastic vibrations. The present paper contains a simple proof of the uniqueness theorem in the case of three-dimensional elasticity, using the same method used earlier to prove uniqueness in the electromagnetic case [I. N. Vekua, Trudy Tbliss. Mat. Inst. **12**, 105–174 (1943); these Rev. **6**, 154]. The boundary condition at infinity is taken to be only

$$\frac{\partial U^{(j)}}{\partial R}-ik_jU^{(j)}=o(R^{-1}), \quad j=1, 2,$$

whereas Kupradze requires in addition that $U^{(j)}=O(R^{-1})$, $j=1, 2$, which is shown to be implied by the first condition (here the displacement vector $U=U^{(1)}+U^{(2)}$, with rot $U^{(1)}=0$ and div $U^{(2)}=0$). *J. B. Diaz.*

Šatašvili, S. H. A spatial problem of the theory of steady elastic vibrations for given displacements on the boundary of the medium. Doklady Akad. Nauk SSSR (N.S.) **83**, 809–811 (1952). (Russian)

The solution of the plane problem of steady vibrations, when the displacements are prescribed on the boundary,

was given by D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 617–622 (1946); these Rev. 8, 361] and the present paper deals with the spatial problem of the theory of elastic vibrations by an analogous method. The boundary value problem in question is reduced to an equivalent system of integral equations of Fredholm type, whose solubility, for almost all values of the frequency parameter ω, is assured by a theorem of J. D. Tamarkin [Ann. of Math. (2) 28, 127–152 (1927)]. *J. B. Diaz* (College Park, Md.).

Satô, Yasuo. Study on surface waves. VI. Generation of Love- and other type of *SH*-waves. Bull. Earthquake Res. Inst. Tokyo 30, 101–120 (1952). (Japanese summary)

L'auteur considère le problème plan (à deux dimensions) de la propagation des ondes transversales dans un milieu élastique indéfini surmonté d'une couche. Il formule d'abord un théorème de réciprocité entre les amplitudes des ondes en deux points situés à grande distance l'un de l'autre et considérés alternativement comme foyer et comme point d'observation. Il considère ensuite une source sinusoïdale cylindrique. Perfectionnant un calcul de Sezawa, il retrouve les ondes coniques réfléchies qui, par superposition, engendrent les ondes de Love à grande distance de la ligne focale. Des tableaux numériques précisent la variation de l'amplitude de ces ondes en fonction de la période.

J. Coulomb (Paris).

Yamamoto, Y. Variational principles of equilibrium of an elasto-plastic body. Quart. Appl. Math. 10, 215–224 (1952).

To establish the desired variational principles, the author assumes the validity of the so-called uniqueness condition. With the help of this relation, it is shown that an expression involving the space integrals of the stress tensor, the body forces, surface forces, and the increment of the stresses due to an admissible displacement is non-negative. This result is called the fundamental principle. By assuming that the final value of the stress increment suffices to determine one of the above integrals, which usually depends upon the stress-strain process, the author is able to express his fundamental principle as a minimum principle. Two formulations of such a minimum principle are given. Upon introducing the involute transformations, these last principles are formulated as maximum principles. Since the integrals which enter into all these principles must be independent of the stress-strain process, no unloading process at the yield condition can be considered. Finally, by expressing these principles in terms of time rate of change instead of increments, the author shows that the Hodge-Prager, Greenberg, Markov-Hill, and Sadowsky principles follow from his general principles. *N. Coburn* (Ann Arbor, Mich.).

Oestreicher, Hans L. Field and impedance of an oscillating sphere in a viscoelastic medium with an application to biophysics. J. Acoust. Soc. Amer. 23, 707–714 (1951).

General linear visco-elastic stress-strain relations are discussed. These are detailed for general stressing of a body represented by Voigt type relations (spring and dashpot in parallel) for both shear and compression properties. These relations are introduced into the equations of motion and particularised for steady state sinusoidal time dependence using the factor $e^{i\omega t}$. The resulting equations are the same as in elasticity theory but with frequency dependent complex elastic constants, so that elastic solutions can be used directly in this field. The analysis of plane waves is given, and also the displacement field for an oscillating rigid sphere in an infinite medium. The impedance to motion of the latter is determined. The analysis is evaluated using constants representing human tissue. *E. H. Lee.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Suzuki, Tatsurō. A differential method to adjust the aberrations of a lens system. Tech. Rep. Osaka Univ. 2, 37–52 (1952).

It is well known that in ray-tracing two useful parameters are the length of the perpendicular from the center of curvature to the ray and the angle this perpendicular makes with the axis. These parameters also prove helpful in finding the rate of change of each image error as the system data are changed. *E. W. Marchand* (Rochester, N. Y.).

Maeder, D. Abbildungsfehler in Cartesischen Linsen-Systemen. Optik 9, 49–74 (1952).

If an optical system consisting of a single refracting or reflecting surface images a single point sharply, the surface is called a Cartesian surface. In the case of an optical system consisting of a number of Cartesian surfaces it is found that the image errors can be expressed in a somewhat simplified form using matrix products. The application to symmetrical systems is considered. *E. W. Marchand.*

Bremmer, H. The derivation of paraxial constants of electron lenses from an integral equation. Appl. Sci. Research B. 2, 416–428 (1952).

L'equazione differenziale delle traiettorie parassiali

$$d^2y/dz^2+k^2(z)y=0$$

viene ricondotta all'equazione integrale

$$y=y_0+y_0'(z-z_0)-\int_{z_0}^{z}k^2(\zeta)(z-\zeta)y(\zeta)d\zeta.$$

Questa viene risolta con il metodo di Liouville-Neumann e se ne deducono i valori degli elementi cardinali del sistema.

G. Toraldo di Francia (Firenze).

Fock, W. A. Fresnelsche Beugung an konvexen Körpern. Abh. Sowjet. Physik 2=Sowjetwissenschaft, Beiheft 28, 7–18 (1951).

Translated from Uspehi Fiz. Nauk 43, 587–599 (1951); these Rev. 13, 514.

Kontorovič, M. I., and Murav'ev, Yu. K. Derivation of the laws of reflection of geometrical optics on the basis of an asymptotic treatment of the diffraction problem. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 22, 394–407 (1952). (Russian)

Il problema della riflessione su una superficie curva viene impostato come un problema di diffrazione e i corrispondenti integrali vengono valutati mediante la nota estensione a due dimensioni del principio della fase stazionaria. Si trovano le leggi dell'ottica geometrica. *G. Toraldo di Francia.*

Lewis, Edward A., and Casey, Joseph P., Jr. Electromagnetic reflection and transmission by gratings of resistive wires. J. Appl. Phys. **23**, 605–608 (1952).

Gli autori calcolano delle curve che permettono di ricavare i coefficienti di riflessione e di trasmissione dei reticoli composti di fili di conduttività finita. L'aumento della resistenza ha per effetto una diminuzione dell'ampiezza del coefficiente di riflessione e della fase del coefficiente di trasmissione. *G. Toraldo di Francia* (Firenze).

Schelkunoff, S. A. Generalized telegraphist's equations for waveguides. Bell System Tech. J. **31**, 784–801 (1952).

Quando una guida d'onda a pareti metalliche è riempita con un dielettrico che non è omogeneo e isotropo, i modi che sussisterebbero nel caso di un dielettrico omogeneo e isotropo risultano in generale accoppiati. L'autore caratterizza l'ampiezza dei vari modi mediante il noto procedimento che fa corrispondere a ciascuno di essi una tensione V_m e una corrente I_m convenzionali e trova delle equazioni che sono generalizzazioni di quella dei telegrafisti. Per esempio

$$\frac{dV_{(m)}}{dz} = -Z_{(m)(s)}I_{(s)} - Z_{(m)[s]}I_{[s]} - {}^{V}T_{(m)(s)}V_{(s)} - {}^{V}T_{(m)[s]}V_{[s]}$$

dove le parentesi tonde e quelle quadre apposte agli indici indicano rispettivamente modi *TM* e *TE* ed è adottata la convenzione tensoriale della sommazione rispetto agl'indici ripetuti. Equazioni simili valgono per $I_{(m)}$, $V_{[m]}$, $I_{[m]}$. L'autore mostra poi come possano calcolarsi i coefficienti di accoppiamento $Z_{(m)(s)}$ ecc. *G. Toraldo di Francia.*

Whitmer, Robert M. Radiation from a dielectric wave guide. J. Appl. Phys. **23**, 949–953 (1952).

La guida considerata è costituita da una lamina dielettrica a facce piane parallele. In essa è immerso un filo percorso da corrente, parallelo alle facce. Questa problema bidimensionale è già stato discusso dall'autore [Proc. I. R. E. **36**, 1105–1109 (1948)]. In questo nuovo lavoro viene valutato l'integral di linea che rappresenta la potenza irraggiata. *G. Toraldo di Francia* (Firenze).

Kleinwächter, Hans. Die einheitliche Berechnung der Strahlungsverteilung des elektrischen und des magnetischen Dipols sowie der Schlitz- und der Spiegelantenne mit Kirchhoffschen Formel. Arch. Elektr. Übertragung **6**, 247–253 (1952).

An expository paper which shows that several different types of antenna problems may be solved by a uniform application of Kirchhoff's formula (Green's theorem). The work appears to be limited to problems involving only rectangular or azimuthally symmetric cylindrical boundaries although it may be extended in a trivial manner to any cylindrical boundary. The technique appears to fail with spherical boundaries and no azimuthal symmetry in which case no component of the electromagnetic field satisfies the scalar wave equation. *W. K. Saunders.*

Haug, Albert. Die Rolle der Ausstrahlungsbedingung bei komplexer Wellenzahl und ihre Bedeutung für das Problem der Oberflächenwelle. Z. Naturforschung **7a**, 501–505 (1952).

The author shows that if one is given a solution u to $(\nabla^2+k^2)u=0$ which satisfies (3a) $\lim_{r\to\infty} r(\partial u/\partial r - iku)=0$ ($e^{-i\omega t}$ time), (3b) ru bounded, and which meets a given boundary condition on a finite smooth surface, then u is uniquely determined. This is true for $k=\alpha+i\beta$, $\beta>0$, as well as $\beta=0$. Moreover, given two regions separated by a smooth surface, each having a different value of k and given continuity of u and $\partial u/\partial n$ in passing through the surface as well as (3a, b), in each of the regions, uniqueness is again assured provided that one $\beta>0$. In particular, the original form of the Sommerfeld surface wave is shown to be consistent with the restriction (3a, b) with one $\beta>0$ and hence not to violate the uniqueness postulate. This appears to settle somewhat the controversy raised by Kahan and Eckart [C. R. Acad. Sci. Paris **226**, 1513–1515 (1948); these Rev. **9**, 637]. See also C. J. Bouwkamp [these Rev. **11**, 143 and Physical Rev. (2) **80**, 294 (1950); these Rev. **12**, 376]. [Reviewer's note: It appears that in the whole space (3b) implies (3a) for $\beta>0$. The extension to the divided space is not evident, but in any event the form of the Sommerfeld surface wave taken by the author satisfies both.] *W. K. Saunders* (Washington, D. C.).

Kothari, L. S., and Bhatnagar, P. L. On a modified definition of Riesz potential and its correspondence to the Wentzel potential. Proc. Nat. Inst. Sci. India **18**, 171–175 (1952).

In a recent paper [Proc. Cambridge Philos. Soc. **47**, 436–442 (1951); these Rev. **12**, 777], Auluck and Kothari gave a modified definition of the Riesz electromagnetic vector-potential in the framework of classical electrodynamics. Whereas the usual definition of the Riesz potential gives on analytical continuation the ordinary Maxwell potential, the modified potential leads to the Wentzel potential. The object of the present note is to discuss in more detail the relation between this modified Riesz potential and the Wentzel potential. *E. T. Copson* (St. Andrews).

Morgan, Samuel P., Jr. Mathematical theory of laminated transmission lines. I. Bell System Tech. J. **31**, 883–949 (1952).

It has been shown by Clogston [Proc. I. R. E. **39**, 767–782 (1951)] that the skin effect loss of a transmission line can be reduced by laminating the conductors into alternate thin layers of conducting and insulating media and filling the space between them with a dielectric that forces the waves to travel along the line with a certain critical velocity. In this paper the author discusses the propagation of electromagnetic waves along two geometrically different laminated lines. One of these is physically unrealizable. It consists of two parallel, plane conductors laminated and separated by a uniform dielectric. The other is of great physical interest. It is a coaxial line with laminated conductors and dielectric filling. The surface impedances of plane and cylindrical conductors are computed and the laminated conductors are replaced by electrically equivalent impedance sheets. Thus the author reduces the problem to one of deducing the propagation properties of an electromagnetic wave in a dielectric bounded by impedance sheets. *C. H. Papas.*

Cramlet, Clyde M. Modular invariants associated with resistance networks. Univ. Washington Publ. Math. **3**, 65–70 (1952).

Invariants for two-circuit and three-circuit connected resistance networks are determined. As developed in a previous paper [Ingram and Cramlet, J. Math. Physics **23**, 134–155 (1944); these Rev. **7**, 403], the transformations for invariance are those associated with all possible choices of independent circuits. As an example of the use of these results, necessary and sufficient conditions for the existence

of three independent circuits with no mutual elements in a three-circuit network are given. *J. Riordan.*

Quantum Mechanics

Fantappiè, Luigi. Determinazione di tutte le grandezze fisiche possibili in un universo quantico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 553–558 (1952).

This presents further discussion by the author of highly general aspects of group invariance of wave equations. *I. E. Segal* (Chicago, Ill.).

Thirring, Walter E. Nichtlineare Terme in Meson-Gleichungen. Z. Naturforschung **7a**, 63–66 (1952).

A very rough calculation based on charged pseudoscalar meson theory of nuclear forces suggests that saturation is due to nonlinear terms. In a footnote, added in proof, the author points out that his results are similar to those of Schiff [Physical Rev. (2) **84**, 1–9, 10–11 (1951)] who, however, does not remark that the needed nonlinear terms arise naturally in quantized field theory. *A. J. Coleman.*

Mirianašvili, M. M. On the relativistic magnetic moment of charged particles. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 613–618 (1947). (Russian)

Margenau [Physical Rev. (2) **57**, 383–386 (1940)] gave two methods of calculating the magnetic moment of a charged particle which agree for non-relativistic particles but disagree for Dirac particles. The author shows that this latter discrepancy can be obviated if the magnetic moment operator $M=e/2mc(L+2S)$ of Margenau's second method is replaced by $M=e/2mc\beta(L+2S)$ where, in the usual notation, β is the fourth Dirac matrix. *A. J. Coleman.*

Šapiro, I. S. On transformation laws for the spinors appearing as solution of the Dirac equation. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 524–538 (1952). (Russian)

It is found that for rest mass $m=0$ there are fourteen, and for $m\neq 0$, four, non-equivalent types of solutions of Dirac's free-particle equation. If A is the 4×4 matrix corresponding to a reflection in the hyperplane orthogonal to the unit vector $\mathfrak{a}$ (in the sense of É. Cartan [Leçons sur la théorie des spineurs, Hermann, Paris, 1938, t. I, p. 56, t. II, p. 9]), then the various types of spinors u transform as follows: $u'=e_1Af_1u$ for $|\mathfrak{a}|^2=1$, and $u'=e_2Af_2u$ for $|\mathfrak{a}|^2=-1$, where (i) for $m\neq 0$, $f_i=1$, $e_i=1$ or $\sqrt{-1}$; (ii) for $m=0$, $e_i=1$ or $\sqrt{-1}$, $f_i=1$ or γ_5 with the two possibilities $f_1=f_2=\gamma_5$, $e_1=e_2=1$ or $\sqrt{-1}$ excluded. These results are applied in a brief discussion of the general form of the interaction hamiltonian for two-particle systems. *A. J. Coleman.*

Matthews, P. T., and Salam, Abdus. The intermediate coupling theory of the pseudoscalar meson-nucleon interaction. Physical Rev. (2) **86**, 715–726 (1952).

This is a bold exploration of the problem of describing the state of a proton or neutron interacting with a pseudoscalar meson field, without using the customary weak- or strong-coupling approximations. The description is based on the "intermediate coupling" approximation of Tomonaga [Progress Theoret. Physics **2**, 6–24 (1947)] which has hitherto been developed only for a completely non-relativistic treatment of the nucleon neglecting recoil. In this paper the nucleon recoil is included approximately, and virtual creation of one nucleon-anti-nucleon pair is taken into account accurately.

The Tomonaga approximation describes the system with a wave-function in which components are present with every possible number of mesons; only when there is more than one meson the wave-function neglects correlations between the configuration of one meson and another. Such a wave-function may be expected to give an accurate over-all picture of the distribution of the meson field. Unfortunately, however, even after making the Tomonaga approximation the equations are still prohibitively complicated, and many other approximations have to be made before explicit solutions can be obtained. It is therefore problematical how far the quantitative results of this paper apply to the real proton-meson system. Qualitatively, the results are probably not far from the truth.

The dimensionless coupling-constant in the pseudo-scalar theory is written $(f^2/4\pi)$. For $f^2/4\pi=10$, a value which is consistent with the experimental evidence, the authors find the following results concerning the field around a proton. At any moment there is a 4% chance of finding one meson in the field, a 13% chance of finding one nucleon-anti-nucleon pair and no meson, an 83% chance of finding no pair and no meson; for all larger numbers of particles the probabilities are of the order of 1% at most.

The importance of these results lies mainly in the strong theoretical support which they give to the subsequent analysis of the same problem by M. Lévy [now in course of publication]. The essential point is that, unexpectedly in view of the large value of the coupling constant, states containing a large number of mesons do not play a significant role in the system. Lévy has analysed the problem using a modified form of weak-coupling theory proposed by Tamm and Dancoff, in which all states involving more than two mesons are simply neglected. Lévy obtained outstandingly good quantitative agreement with the experimental properties of the proton-neutron system. The present paper explains why such good agreement was to be expected, and enables an estimate to be made of the errors still inherent in Lévy's treatment. *F. J. Dyson* (Ithaca, N. Y.).

Suura, Hiroshi, Mimura, Yoichi, and Kimura, Toshiei. On the analytic behaviour of Dyson transformation function. Progress Theoret. Physics **7**, 171–184 (1952).

Consider a quantum-mechanical system described in the interaction representation by an interaction Hamiltonian $H(t)$, the corresponding system without interaction being defined in the Schrödinger representation by a free-particle Hamiltonian H_0. Let Φ_a be a stationary state of the non-interacting system with energy E_a. Then a stationary state Ψ_a of the interacting system with the same energy is defined by the unitary transformation

$$\Psi_a(t)=U(t)\Phi_a, \tag{1}$$

$$U(t)=1+\sum_{n=1}^{\infty}(-i)^n\int_{-\infty}^{t}H(t_1)dt_1\cdots\int_{-\infty}^{t_{n-1}}H(t_n)dt_n, \tag{2}$$

provided that the series and the integrals in (2) are well-defined and convergent [B. Ferretti, Nuovo Cimento (9) **8**, 108–131 (1951); these Rev. **12**, 890]. In general, however, the interacting and non-interacting systems have different energy spectra, hence a unitary transformation (1) cannot exist, hence the series (2) cannot always converge.

The authors investigate the behaviour of (2) by considering $U(t)$ to be defined as the limit as $\epsilon\to 0$ of the operator

$U(t, \epsilon)$, the latter being obtained by replacing $H(t)$ by $H(t) \exp(\epsilon t)$ in (2). For finite ϵ, $U(t, \epsilon)$ is an operator describing the effect on a non-interacting system of switching on the interaction $H(t)$ gradually. The authors assume that the series for $U(t, \epsilon)$ is itself convergent for all positive ϵ. They also assume that the interacting system has stationary states which can be separated clearly into bound states with $E_a<0$ and continuum states with $E_a>0$. Under these assumptions, it is shown that for every state Φ_a with $E_a>0$ the limit $U(t)\Phi_a$ exists and defines a stationary continuum state of the interacting system. But the convergence of $U(t, \epsilon)\Phi_a$ to $U(t)\Phi_a$ is not uniform for $E_a>0$. For every positive ϵ, there is a range of states Φ_a, with E_a of the order of magnitude ϵ, such that $U(t, \epsilon)\Phi_a$ contains a linear combination of the bound states of the interacting system. This phenomenon makes clear exactly how the definition (2) of the transformation operator $U(t)$ breaks down for systems involving bound states.

While the results of this paper are illuminating, the presentation is in some respects misleading. The authors write as if their analysis applies immediately to a quantized field theory. They assert the convergence of the series $U(t, \epsilon)$ as if it were a fact and not an assumption. In field theory, the series $U(t, \epsilon)$ is probably always divergent, and the spectrum of bound and unbound states is much more complicated than that postulated in this paper. Strictly, the analysis of this paper applies only to a simple two-particle system without quantized fields. *F. J. Dyson.*

Schönberg, M. On the general theory of damping in quantum mechanics. Nuovo Cimento (9) **8**, 817–842 (1951).

This is the fourth in a series of papers which aim at a possibly general treatment of quantum theoretical perturbation problems [Nuovo Cimento (9) **8**, 243–270, 403–431, 651–682 (1951); these Rev. **13**, 193, 607]. A generalization of the integral equation of the quantum theory of radiation damping which applies to cases with both continuous and discrete levels, has already been derived in the second paper. Now the theory of damping is formulated with the aid of the generalized resolvent operators $R(\lambda, t)$ which have been introduced in the third paper. In the stationary case these operators fulfill the equation $(H-\lambda)R(\lambda)=1$, where H is the total Hamiltonian. The integral equation of the damping theory, in the Heitler-Peng form [Proc. Cambridge Philos. Soc. **38**, 296–312 (1942)], is shown to be an equation for the determination of the branch R_+ of the resolvent which is defined by $R_+(\lambda)=R(\lambda)$ for $\mathrm{Im}\,\lambda>0$. The method applies to cases in which it is not possible to set up a correspondence between unperturbed and perturbed stationary states. Generalization of the method to the non-stationary case is achieved by introducing additional operators. A width operator $\Gamma_+(\lambda)$ is defined in terms of R; formulas, previously given by Heitler and Ma [Proc. Roy. Irish Acad. Sect. A. **52**, 109–125 (1949); these Rev. **11**, 298] in their non-stationary theory of radiation damping for discrete states, follow from this definition. Finally, the proposed treatment is shown to be just as general as Schwinger's [Physical Rev. **74**, 1439–1461 (1948); these Rev. **10**, 345] which, however, applies only to the stationary case; Fukuda and Miyazima's [Progress Theoret. Physics **5**, 849–860 (1950); these Rev. **12**, 785] derivation of the damping equations from Schwinger's theory is confirmed. Hence, the proposed method represents a unification of the Heitler-Peng and Schwinger theories.

E. Gora (Providence, R. I.).

Schoch, Arnold, und Steinwedel, Helmut. Zum Energie-Impuls-Tensor linearer klassischer Feldtheorien. Z. Naturforschung **7a**, 66–70 (1952).

Chiefly concerned with formal developments related to Feynman's approach to classical field theory [Physical Rev. (2) **74**, 939–946 (1948); these Rev. **10**, 222]. Explicit formulae are given for the energy-momentum tensor $T_\nu^\mu(\eta)$ in terms of the Pauli regulator parameter η, in the case of vector and scalar fields. *A. J. Coleman* (Toronto, Ont.).

Polievktov-Nikoladze, N. M. On the derivation of the relativistic equations for free particles with spin 1/2. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 11–17 (1948). (Russian)

Consider any wave-equation satisfying the following four postulates. (1) It is invariant under the full Lorentz group. (2) It is linear and homogeneous in the wave-function ψ. (3) The sum of the squares of the components of ψ can be consistently interpreted as a probability-density. (4) The equation contains time-derivatives only to first order. The author proves that such a wave-equation is necessarily of the form

$$(i(\partial/\partial x_4)+\gamma_\alpha(\partial/\partial x_\alpha))\psi=(i/hc)\gamma_0\psi,$$

where the γ_α $(\alpha=1, 2, 3)$ and γ_0 are $(4r\times 4r)$ matrices satisfying the commutation rules

$$\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\delta_{\alpha\beta}, \quad \gamma_\alpha\gamma_0+\gamma_0\gamma_\alpha=0.$$

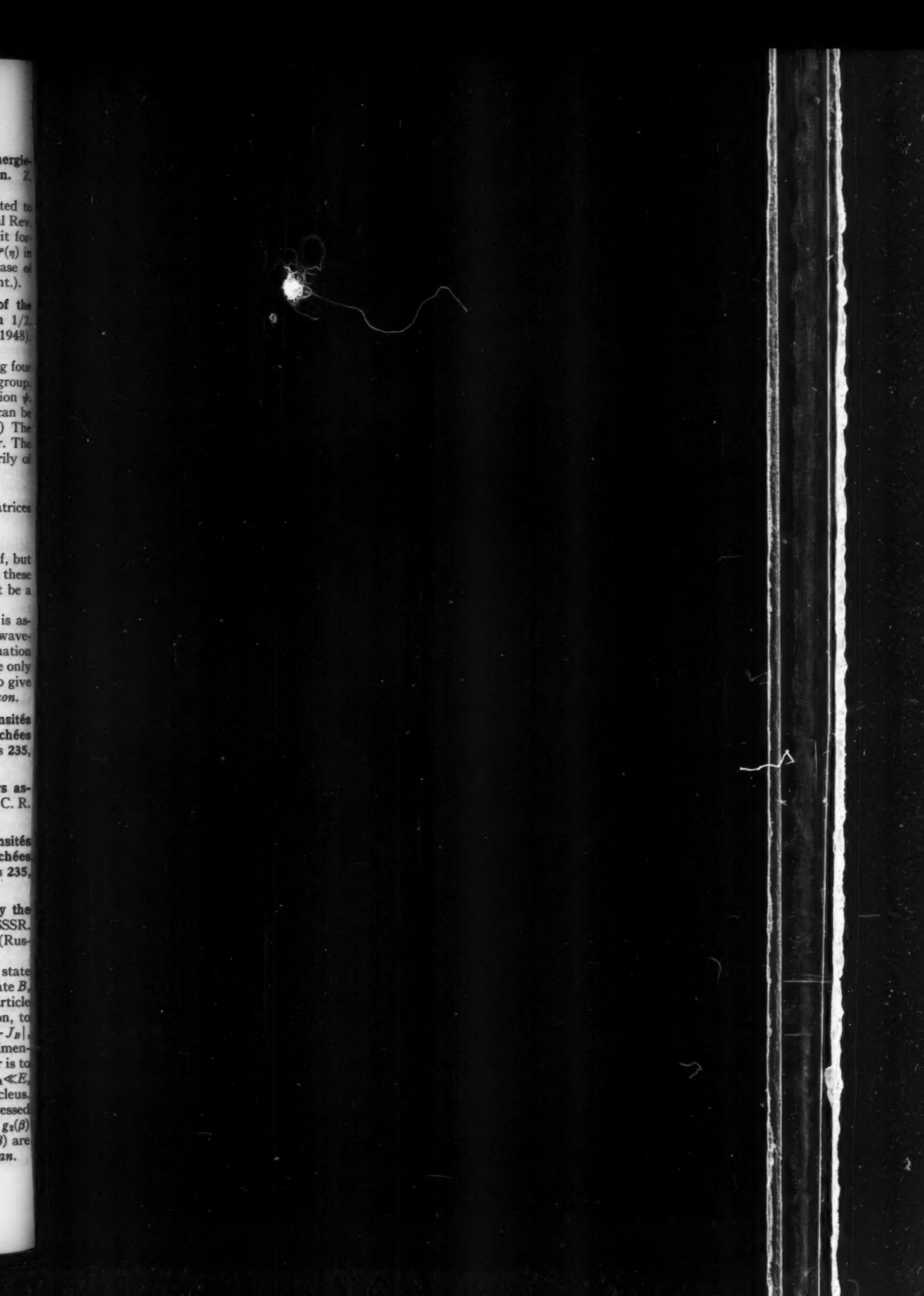
The equation describes a particle having spin one-half, but with possibly several different values of its rest-mass, these being given by the eigenvalues of γ_0^2 which need not be a multiple of the unit matrix.

In the usual derivation of the Dirac equation, it is assumed, in addition to postulates (1)–(4), that the wave-equation is consistent with the second-order equation $(\partial^2/\partial x_\alpha^2)\psi=k^2\psi$. The author's argument shows that the only effect of this very restrictive additional postulate is to give the particle a unique rest-mass. *F. J. Dyson.*

Winogradzki, Judith. Sur la forme spinorielle des densités de valeur moyenne des grandeurs physiques attachées aux particules de spin 1/2. C. R. Acad. Sci. Paris **235**, 463–465 (1952).

Winogradzki, Judith. Sur huit familles d'opérateurs associés aux observables des particules de spin 1/2. C. R. Acad. Sci. Paris **235**, 505–507 (1952).

Winogradzki, Judith. Sur les relations entre les densités de valeur moyenne des grandeurs physiques attachées aux particules de spin 1/2. C. R. Acad. Sci. Paris **235**, 533–535 (1952).

Ter-Martirosyan, K. A. The excitation of nuclei by the Coulomb field of charged particles. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 284–296 (1952). (Russian)

The total cross-section, for a nuclear transition from state A, of total angular momentum J_A and energy E_A, to state B, due to Coulomb interaction with a heavy charged particle of energy E is found, by a "semi-classical" calculation, to have the form $\sum c_n g_n(\beta)$ where $|J_A+J_B|\geqq n\geqq|J_A-J_B|$, c_n depends on the nuclear states A and B, and β is a dimensionless impact parameter. The chief point of the paper is to study the functions $g_n(\beta)$ in the cases (a) $E_B-E_A\ll E$, (b) $E_B-E_A\sim E$, (c) $E>$Coulomb barrier of the nucleus. Only in the case $n=1$ (dipole transition) is $g_n(\beta)$ expressed in terms of known functions. Graphs are given for $g_2(\beta)$ and $g_3(\beta)$ in case (a), and asymptotic formulas for $g_n(\beta)$ are obtained for $\beta\to 0$, and $\beta\gg 1$. *A. J. Coleman.*